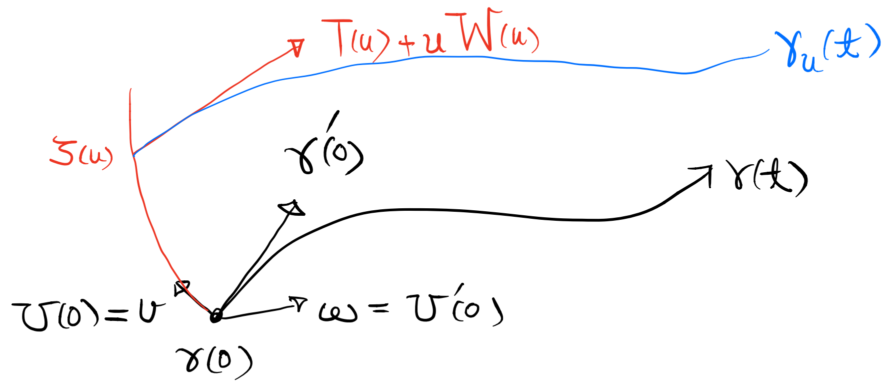
Pf of Lemma 2

($\Leftarrow$) Proved in previous chapter.

($\Rightarrow$) Let U be a Jacobi field along γ with

$$\begin{cases} U(0) = v \\ U'(0) = w \end{cases} \quad \left(\text{by identifying } T_{\vec{p}}(T_{\gamma(0)}M) \cong T_{\gamma(0)}M \right)$$



Let $\xi : [0,\varepsilon] \to M$ be a geodesic such that

$$\xi(0) = \gamma(0) \quad \text{and} \quad \xi'(0) = v$$

Define parallel vector fields $T(u)$ and $W(u)$ for $u \in [0,\varepsilon]$ along ξ such that $T(0) = \gamma'(0)$ and $W(0) = w$;

$$\Gamma(t,u) = \gamma_u(t) = \exp_{\xi(u)}\left[t\left(T(u) + uW(u)\right)\right] \quad \forall\, u \in [0,\varepsilon]$$

Let U_1 = transversal vector field of γ_u along $\gamma = \gamma_0$.

Then U_1 is a Jacobi field.

And $U_1(0) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma_u(0)$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \exp_{\zeta(u)}(0)$$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \zeta(u)$$

$$= \zeta'(0) = v.$$

Since $T_1 = d\Gamma\left(\frac{\partial}{\partial t}\right)$ is a vector field along Γ & when restricted to γ, we have

$$[T_1, U_1] = 0.$$

Hence $U_1'(0) = D_{\gamma'(0)} U_1 = D_{U_1(0)} T_1 \quad (\text{since } [T_1, U_1] = 0)$

$$= D_v T_1 = D_{\zeta'(0)} T_1$$

Note that $T_1(\zeta(u)) = \frac{\partial}{\partial t}\Big|_{t=0} \exp_{\zeta(u)}[t(T(u) + u W(u))]$

$$= T(u) + u W(u)$$

$\Rightarrow U_1'(0) = D_{\zeta'(0)} T_1 = D_{\zeta'(0)}[T(u) + u W(u)]$

$$= W(0) \quad (\text{Since } D_{\zeta'} T = D_{\zeta'} W = 0)$$

$$= w.$$

Altogether $U(0) = U_1(0)$ & $U'(0) = U_1'(0)$

Uniqueness of Jacobi field $\Rightarrow U = U_1 =$ transversal vector field $\#$

<u>Lemma 3</u> Let U be a Jacobi field along a geodesic γ.

Then $\exists$ constants a, b such that

$$U = U^{\perp} + (at+b)\gamma',$$

where $U^{\perp}$ is a Jacobi field s.t. $\langle U^{\perp}, \gamma' \rangle = 0$, $\forall t$.

<u>Pf</u>: Consider

$$\frac{d^2}{dt^2}\langle U, \gamma' \rangle = \frac{d}{dt}\left(D_{\gamma'} \langle U, \gamma' \rangle\right)$$

$$= \frac{d}{dt}\left(\langle D_{\gamma'}U, \gamma' \rangle + \langle U, \overset{0}{\cancel{D_{\gamma'}\gamma'}} \rangle\right)$$

$$= \langle U'', \gamma' \rangle$$

$$= -\langle R_{\gamma' U}\gamma', \gamma' \rangle = 0$$

$\Rightarrow \quad \langle U, \gamma' \rangle = \tilde{a}t + \tilde{b}$ for some constants $\tilde{a}, \tilde{b}$.

Let $U^{\perp} = U - \langle U, \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|}$

$$= U - \left(\frac{\tilde{a}}{|\gamma'|^2}t + \frac{\tilde{b}}{|\gamma'|^2}\right)\gamma'$$

Since $|\gamma'| \equiv$ const.

$$U^\perp = U - (at+b)\gamma'$$

with $\quad a = \dfrac{\tilde{a}}{|\gamma'|^2}, \quad b = \dfrac{\tilde{b}}{|\gamma'|^2}$ are constants,

and satisfies $\langle U^\perp, \gamma' \rangle = 0$.

$$(U^\perp)'' = U'' - [(at+b)\gamma']''$$

$$= U'' = -R_{\gamma' U}\gamma'$$

$$= -R_{\gamma' U^\perp}\gamma' - (at+b)\cancelto{0}{R_{\gamma'\gamma'}}\gamma'$$

$\therefore$ $U^\perp$ is a Jacobi field. ※

<u>Lemma 4</u>: If U is a Jacobi field along a geodesic γ such that $\langle U(t_1), \gamma'(t_1) \rangle = \langle U(t_2), \gamma'(t_2) \rangle = 0$ for 2 different t_1 & t_2. Then $\langle U(t), \gamma'(t) \rangle = 0$, $\forall t$.

(Pf: Since $\langle U(t), \gamma'(t) \rangle$ is linear in t.)

<u>In summary</u>, we have the following facts of Jacobi fields.

(A) Let $\zeta : [0, \varepsilon] \to M$ be a curve in M,

$$u \mapsto \zeta(u)$$

$T(u)$, $W(u)$ parallel vector fields along ζ.

Then $$\boxed{\gamma_u(t) = \exp_{\xi(u)}\left[t\left(T(u) + uW(u)\right)\right]}$$

determines a 1-para family of geodesic $\{\gamma_u\}$ s.t. its transversal vector field $U(t)$ along γ_0 is a Jacobi field with $\begin{cases} U(0) = \xi'(0) \\ U'(0) = W(0) \end{cases}$.

(B) [If we take $\xi(u) \equiv x \in M$ (const. curve) in (A)]

$\forall\, x \in M$; $T, w \in T_xM$. Then the 1-para. family of geodesics $\{\gamma_u\}$ defined by

$$\boxed{\gamma_u(t) = \exp_x\left[t(T + uw)\right]}$$

has a transversal vector field $U(t)$ s.t.

$U(t)$ is a Jacobi field with $\begin{cases} U(0) = 0 \\ U'(0) = w \end{cases}$

(C) [Furthermore, adding condition $\langle T, w\rangle = 0$ to (B)]

Let $x \in M$; $T, w \in T_xM$ s.t. $\langle T, w\rangle = 0$.

Let $$\gamma_u(t) = \exp_x\left[t(T + uw)\right]$$

Then the transversal vector field $U(t)$ of $\{\gamma_u\}$ is a normal Jacobi field with $\begin{cases} V(0)=0 \\ V'(0)=w \end{cases}$

(here: normal Jacobi field = Jacobi field normal to the geodesic)

Pf of (C)

We need

Lemma 5 (Gauss lemma)

Let M be complete, $x \in M$, $\vec{p} \in T_xM$,

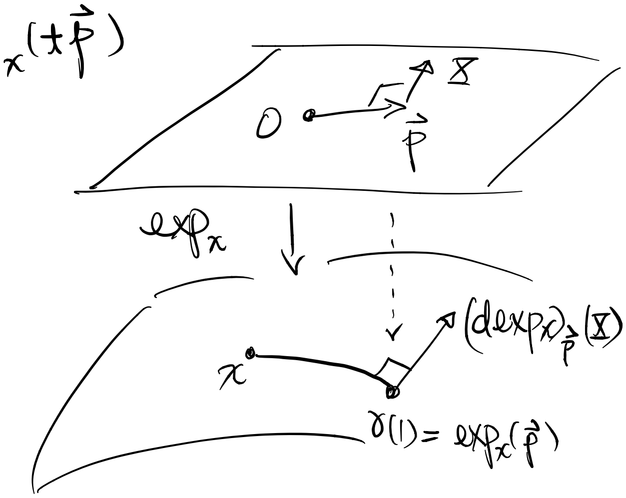
$$X \in T_{\vec{p}}(T_xM) \cong T_xM.$$

If $\langle \vec{p}, X \rangle = 0$, then

$$\langle (d\exp_x)_{\vec{p}}(X), \gamma'(1) \rangle = 0$$

where $\gamma : [0,1] \to M$

$$t \mapsto \exp_x(t\vec{p})$$



Pf: Let $\xi : [0,\varepsilon] \to T_xM$ be a curve in T_xM s.t.

$\xi(0) = \vec{p}$, $\xi'(0) = X$;

and that $\xi([0,\varepsilon]) \subset S^{n-1}_{|\vec{p}|} \subset T_xM$

Such ξ exists since $X \perp \vec{p} \Rightarrow X \in T_{\vec{p}} S^{n-1}_{|\vec{p}|}$.

Consider $\Gamma : [0,1] \times [0,\varepsilon] \to M$

$$(t,u) \longmapsto \exp_x[t\,\xi(u)]$$

Let $T = d\Gamma\left(\frac{\partial}{\partial t}\right)$ and $U = d\Gamma\left(\frac{\partial}{\partial u}\right)$.

Then $\gamma(t) = \Gamma(t,0)$,

$\gamma'(1) = T(\gamma(1))$

$(d\exp_x)_{\vec{p}}(X) = U(\gamma(1))$

Since $|\xi(u)| = |\vec{p}|$, we have $\langle T, T\rangle = |\vec{p}|^2$

(geodesic has const. speed)

$$\therefore\ T\langle U, T\rangle = \langle D_T U, T\rangle + \langle U, \cancelto{0}{D_T T}\rangle \qquad (\gamma = \text{geodesic})$$

$$= \langle D_U T, T\rangle \qquad ([U,T] = 0)$$

$$= \frac{1}{2} U\langle T, T\rangle = 0.$$

$$\Rightarrow \langle U, T\rangle = \text{constant along } \gamma$$

$$= \lim_{t\to 0} \langle U(t), T(t)\rangle = \langle \cancelto{0}{U(0)}, T(0)\rangle$$

$$= 0. \quad \#$$

<u>Pf of (C)</u>: Let $\xi : [0,\varepsilon] \to T_x M$

$$u \longmapsto t(T + uW)$$

Then $\langle \zeta'(0), \zeta(0) \rangle = \langle t\omega, tT \rangle = t^2 \langle \omega, T \rangle \overset{\text{by assumption}}{=} 0$

and $(d\exp_x)_{(tT)}(\zeta'(0)) = U(t)$ $\left(\begin{array}{l} U = \text{transversal vector} \\ \text{field of } \exp_x[t(T+u\omega)] \end{array} \right)$

Consider the curve
$$\gamma : [0,1] \to M$$
$$\tau \mapsto \exp_x(\tau(tT))$$

Note that $\gamma_0(t) = \exp_x(tT)$ of the family $\exp_x[t(T+u\omega)]$

$$\Rightarrow \gamma'(1) = \frac{d}{d\tau}\Big|_{\tau=1} \left(\exp_x(\tau(tT)) \right) = (d\exp_x)_{(tT)}(tT)$$
$$= t\,(d\exp_x)_{(tT)}(T)$$
$$= t\,\gamma_0'(t). \qquad (\text{" ' " means derivatives wrt } t)$$

Applying the Gauss lemma to $\gamma(\tau)$ and $X = \zeta'(0) = t\omega$,

$$\left(\vec{p} = \zeta(0) = tT,\ \langle X, \vec{p} \rangle = \langle \zeta'(0), \zeta(0) \rangle = 0 \right)$$

we have
$$\langle U(t), \gamma_0'(t) \rangle = \langle (d\exp_x)_{(tT)}(\zeta'(0)),\ \tfrac{1}{t}\gamma'(1) \rangle$$
$$= \frac{1}{t} \langle (d\exp_x)_{(tT)}(\zeta'(0)), \gamma'(1) \rangle = 0$$

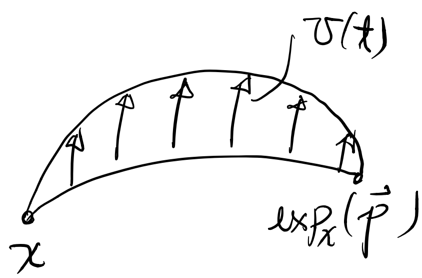
$\Rightarrow U$ is normal. #

§6.2 Cartan-Hadamard Theorem

Lemma 6 $(d\exp_x)_{\vec{p}}$ is singular

$\Leftrightarrow$ $\exists$ normal Jacobi field $V(t)$ along $\gamma(t) = \exp_x(t\vec{p})$, not identically zero

s.t. $V(0) = V(1) = 0$



Pf: By the lemma right before the original version of Gauss lemma in Ch4, $(d\exp_x)_{\vec{p}}$ is non-degenerate in the direction of $\vec{p}$. Therefore, we only need to consider X s.t. $\langle X, \vec{p}\rangle = 0$.

Let $X \in T_xM \cong T_{\vec{p}}(T_xM)$ s.t. $\langle X, \vec{p}\rangle = 0$

Then $\gamma_u(t) = \exp_x[t(\vec{p} + uX)]$

gives a normal Jacobi field with $V(0) = 0$

and $V'(0) = X$ (by fact (C))

Furthermore $V(1) = (d\exp_x)_{\vec{p}}(X)$.

Therefore, if $\underline{X} \in \ker (d\exp_x)_{\vec{p}}$ & $\underline{X} \neq 0$,

then $U(t)$ is a non-identically zero normal Jacobi field with $U(0) = U(1) = 0$. This proves the direction "$\Rightarrow$".

Conversely, any normal Jacobi field is the transversal vector field of a 1-para family of geodesics given by

$$\gamma_u(t) = \exp_{\xi(u)} [t(T(u) + uW(u))]$$

with $\xi(0) = \gamma(0)$, $\xi'(0) = U(0)$, &

T, W = parallel vector fields along $\xi(u)$, & $\langle T, W \rangle = 0$

Since $U(0) = 0$, we may take $\xi(u) \equiv \gamma(0) = x$, $T = \vec{p}$ & $W = \underline{X} = U'(0) \neq 0$ & then $\langle T, W \rangle = 0$.

Therefore
$$0 \underset{\text{assumption}}{=} U(1) = (d\exp_x)_{\vec{p}}(\underline{X})$$

i.e. $0 \neq \underline{X} \in \ker ((d\exp_x)_{\vec{p}})$

$\therefore$ $(d\exp_x)_{\vec{p}}$ is singular. ※

Def: If $(d\exp_x)_{\vec{p}}$ is singular, then $\vec{p}$ is called a conjugate point of the map $\exp_x$, and $\exp_x(\vec{p})$ is called a conjugate point of x along the geodesic $\gamma(t) = \exp_x(t\vec{p})$.

Thm 7 (Cartan-Hadamard)

(1) Let M be a complete Riemannian mfd with nonpositive sectional curvature. Then $\forall\, x \in M$, $\exp_x : T_xM \to M$ has no conjugate point.

(2) If M is a simply-connected complete Riem. mfd. s.t. for some $x \in M$, $\exp_x : T_xM \to M$ has no conjugate point, then $\exp_x : T_xM \to M$ is a diffeomorphism.

Pf of (1): Let U be a normal Jacobi field with $U(0)=0$ along a geodesic $\gamma : [0,\infty) \to M$. ($M$ = complete)

Let $f(t) = \langle U(t), U(t) \rangle$ along γ, then

$$f'(t) = 2\langle U', U \rangle$$

$$\Rightarrow \quad f''(t) = 2\langle V', V'\rangle + 2\langle V'', V\rangle$$
$$= 2|V'|^2 - 2\langle R_{\gamma' V}\gamma', V\rangle$$

Since $\langle R_{\gamma' V}\gamma', V\rangle = K(\mathrm{span}\langle\gamma', V\rangle)\,|\gamma' \wedge V|^2$
$$= K|\gamma'|^2|V|^2 \le 0 \quad (\text{since } \langle\gamma', V\rangle = 0),$$

$$f''(t) \ge 0, \quad \forall t \in [0,\infty).$$

Now suppose $\gamma(t_0)$ is a conjugate point of x along some geodesic $\gamma : [0,\infty) \to M$. Then lemma 6 $\Rightarrow \exists$ non-trivial ($V(t) \not\equiv 0$ on $[0,t_0]$) normal Jacobi vector field $V(t)$ along γ s.t.

$$V(0) = V(t_0) = 0.$$

Applying the above, $|V(t)|^2$ is convex in t

$$\Rightarrow \quad 0 \le |V(t)|^2 \le \max\{|V(0)|^2, |V(t_0)|^2\} = 0,$$
$$\forall t \in [0, t_0]$$

$\Rightarrow V \equiv 0$ on $[0,t_0]$. Contradiction. #

The proof of (2) is much longer and we need the following lemmas (8 & 9):

<u>Lemma 8</u>: Let $\varphi : M \to N$ be a <u>local</u> isometry between (connected)

Riemannian manifolds M & N. If M is <u>complete</u>, then N is complete and φ is a covering map.

<u>Pf</u>: <u>Step 1</u> : <u>φ is surjective and complete</u>

- "φ = local isometry" $\Rightarrow$ $\varphi(M)$ open in N.
- Suppose $\gamma \subset N$ is a geodesic such that
  $\gamma \cap \varphi(M) \neq \emptyset$.

Then $\exists\, x \in M$ such that $\varphi(x)$ is a point on γ.

Since φ is a local isometry, then near the point x, $\varphi^{-1} \circ \gamma$ defines a geodesic segment in a nbd. of x in M (passing thro the point x).

The completeness of M implies $\varphi^{-1} \circ \gamma$ extends to a geodesic $\tilde{\gamma} \subset M$ defined on $(-\infty, \infty)$.

By assumption on φ, we have $\varphi \circ \tilde{\gamma} : (-\infty, \infty) \to \varphi(M) \subset N$ is a geodesic on M passing thro. $\varphi(x)$, and in a nbd of $0 \in (-\infty, \infty)$, $\varphi \circ \tilde{\gamma} = \varphi \circ (\varphi^{-1} \circ \gamma) = \gamma$.

In particular, $(\varphi \circ \tilde{\gamma})'(0) = \gamma'(0)$.

Therefore, uniqueness of geodesic $\Rightarrow$ $\varphi \circ \tilde{\gamma} = \gamma$

$\therefore$ $\gamma \subset \varphi(M)$.

So we've proved that if a geodesic segment γ in N intersects $\varphi(M)$, then $\gamma \subset \varphi(M)$, and extends to $(-\infty, \infty)$.

Now suppose y is a limiting point of $\varphi(M)$ in N, then $\exists\, x \in M$ and $\exists$ a geodesic $\gamma(t)$, $t \in [0,1]$, in N such that $\gamma(0) = \varphi(x)$ and $\gamma(1) = y$.

Therefore, by the above argument, $y = \gamma(1) \subset \varphi(M)$

$\therefore$ $\varphi(M)$ is closed in N.

Hence $\varphi(M)$ is both open and closed (non-empty) in a connected manifold N, we have $\varphi(M) = N$

$\Rightarrow$ φ is <u>surjective</u>.

Note that, we've in fact proved the following

- commutative diagram:

$$\begin{array}{ccc} T_xM & \xrightarrow{d\varphi} & T_{\varphi(x)}N \\ {\scriptstyle \exp_x^M}\downarrow & \circlearrowleft & \downarrow {\scriptstyle \exp_{\varphi(x)}^N} \\ M & \xrightarrow{\varphi} & N \end{array} \quad \text{(local isom)}$$

- and N is complete.

Even more: (for $\delta > 0$ small s.t. $\exp_x$ is a diffeo when restricted to a ball of radius δ) we have

$$\begin{tikzcd} B^M(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp_x^M"'] \arrow[dr, phantom, "\circlearrowleft"] & B^N(\delta) \arrow[d, "\exp^N_{\varphi(x)}"] \\ B^M_\delta(x) \arrow[r, "\varphi"] & B^N_\delta(\varphi(x)) \end{tikzcd}$$

(Ex!).