§4.2 Gauss Lemma, minimizing geodesic

Let (M,g) = Riemannian manifold,

$x \in M$, & $\delta > 0$ sufficiently small such that

$\exp_x : B(\delta) \to B_\delta$ is a diffeomorphism,

where $B(\delta) = \{ v \in T_xM : |v| < \delta \}$, $\left(|v| = \langle v, v\rangle^{1/2}\right)$

$B_\delta = \exp_x(B(\delta))$.

Then • $\gamma(t) = \exp_x(tv)$, $t \in (0,1]$, $v \in B(\delta)$ is called a radial geodesic (segment) joining x to $\exp_x(v)$.

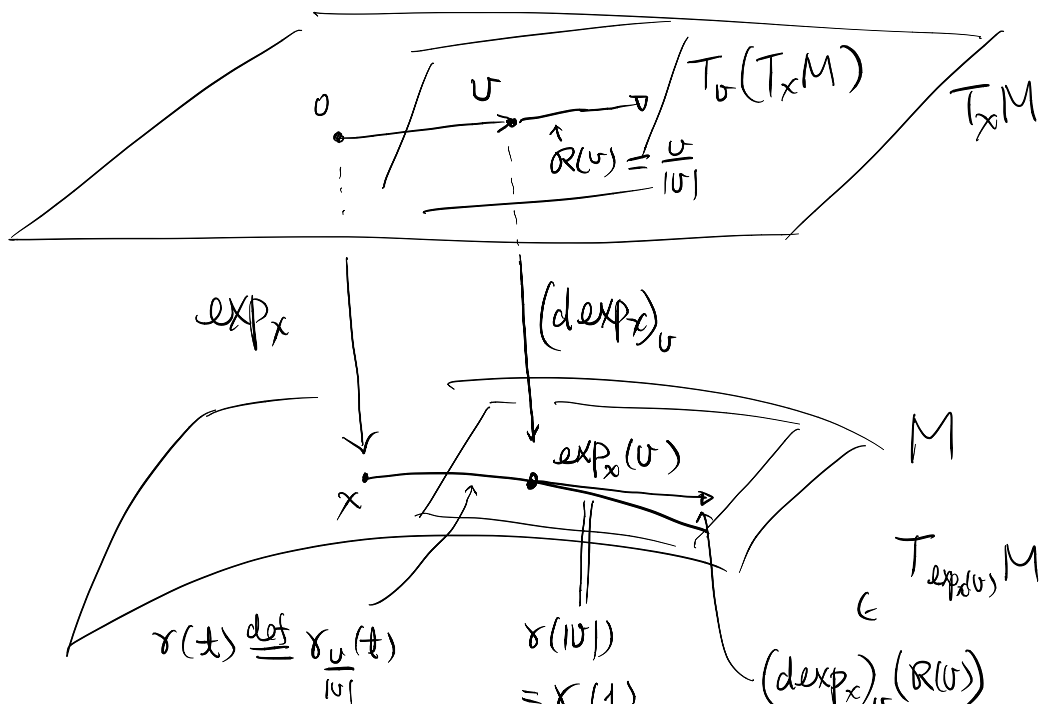
And $\forall t \in (0,\delta)$,

- $S_t = \exp_x(\{ v \in T_xM : |v| = t \})$ is called the geodesic sphere of radius t centered at x.
- $B_t = \exp_x(B(t))$ is called the geodesic ball of radius t centered at x.

Lemma : (M,g), x, δ as above. Define a vector field R on $T_xM \setminus \{0\}$ by

$$\mathcal{R}(v) = \frac{v}{|v|} \quad \left(\begin{array}{r} \mathcal{R} : T_xM\setminus\{0\} \longrightarrow T(T_xM\setminus\{0\}) \\ \text{with } T_v(T_xM\setminus\{0\}) \cong T_xM \end{array} \right)$$

Then $\left| (d\exp_x)_v (\mathcal{R}(v)) \right| = 1 \qquad (\text{for } v \in B(\delta))$



Pf : For $v \in B(\delta)\setminus\{0\} \subset T_xM\setminus\{0\}$, let

$\gamma(t) = \gamma_{\frac{v}{|v|}}(t)$ the normalized geodesic on M

such that $\begin{cases} \gamma(0) = x \\ \gamma'(0) = \frac{v}{|v|}. \end{cases}$ $\left(\text{i.e. } \gamma(t) = \exp_x\left(t\frac{v}{|v|}\right)\right)$

By definition of $\exp_x$,

$$\exp_x(v) = \gamma(|v|)$$

Since $R(v)$ = unit tangent vector of the line

$$t \mapsto v + t\,R(v),$$

$$(d\exp_x)_v(R(v)) = \frac{d}{dt}\Big|_{t=0} (\exp_x)(v + tR(v))$$

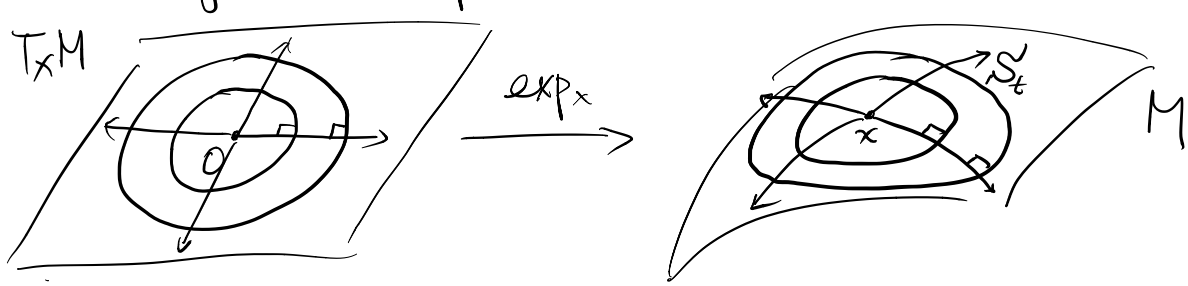
$$= \frac{d}{dt}\Big|_{t=0} (\exp_x)\left((|v|+t)\frac{v}{|v|}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma(|v|+t)$$

$$= \gamma'(|v|)$$

$$\therefore \left|(d\exp_x)_v(R(v))\right| = |\gamma'(|v|)| = |\gamma'(0)| = 1 \quad \#$$

Gauss Lemma: Radial geodesics are orthogonal to the geodesic spheres S_t, $\forall\, t \in (0, \delta)$.



Pf : Define a diffeo

$$F : \underset{\substack{\cup\\ (p,t)}}{S^{n-1}\times(0,\delta)} \overset{\subset T_xM\setminus\{0\}}{} \longrightarrow \underset{\substack{\cup\\ F(p,t)=\exp_x(tp)}}{B_\delta\setminus\{x\}}$$

$$(p,t) \longmapsto F(p,t) = \exp_x(tp)$$

Then for fixed $t\in(0,\delta)$,

$$F(\cdot,t) : S^{n-1}\times\{t\} \longrightarrow S_t$$

is a diffeomorphism.

Let γ = radial geodesic intersecting S_t at the point $\exp_x(tp)$.

We take a local coordinate $\{y^1,\cdots,y^{n-1}\}$ around $p\in S^{n-1}$. And let r be the natural parameter of the interval $(0,\delta)$.

Then
$$\begin{cases} R = dF\left(\frac{\partial}{\partial r}\right) \\ Y_i = dF\left(\frac{\partial}{\partial y^i}\right) \end{cases}$$

are vector fields on $B_\delta\setminus\{x\}\subset M$ such that Y_i are tangential to S_t and form a basis of T_yS_t for $y\in S_t\subset B_\delta\setminus\{x\}$, and

R is tangential to a radial geodesic.

Therefore, we need to show that

$\langle R, Y_i \rangle = 0$, $\forall\, i = 1, \cdots, n-1$, at $\exp_x(tp)$.

Consider $\langle R, Y_i \rangle$ along the radial geodesic γ.

derivative wrt parameter r.

Then $\langle R, Y_i \rangle' = R \langle R, Y_i \rangle$

$= \langle D_R R, Y_i \rangle + \langle R, D_R Y_i \rangle$

$= 0 + \langle R, D_{Y_i} R \rangle + \langle R, [R, Y_i] \rangle$

(Since $D_R R = D_{\gamma'} \gamma' = 0$)

However $[R, Y_i] = [dF(\frac{\partial}{\partial r}), dF(\frac{\partial}{\partial y_i})]$

$= dF([\frac{\partial}{\partial r}, \frac{\partial}{\partial y_i}])$

$= 0$

$\therefore \langle R, Y_i \rangle' = \langle R, D_{Y_i} R \rangle = \frac{1}{2} Y_i \langle R, R \rangle$

$= \frac{1}{2} Y_i (1)$ (by lemma)

$= 0$

$\Rightarrow \langle R, Y_i \rangle = \lim_{r \to 0} \langle R, Y_i \rangle (\gamma(r)) = 0$ since $|Y_i| \to 0$ as $\gamma(r) \to x$ #

Thm: Let • (M,g) = Riemannian manifold

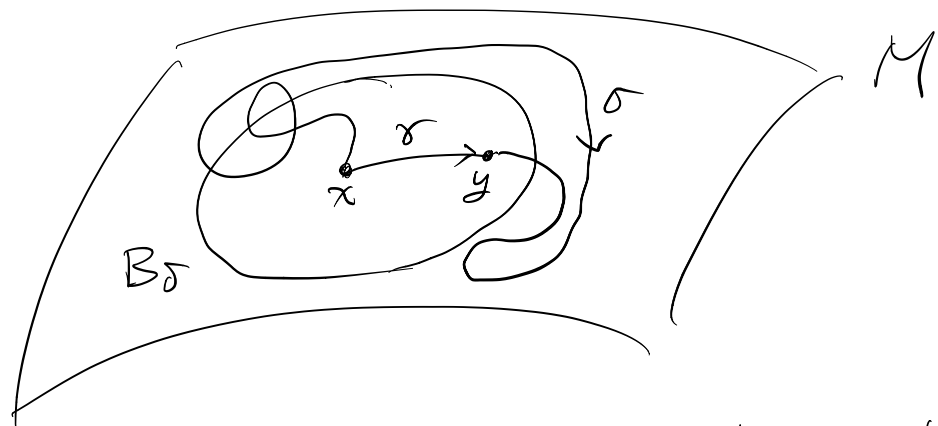
• $x \in M$

• $\delta > 0$ s.t. $\exp_x : B(\delta) \to B_\delta$ is a diffeo.

• γ = unique radial geodesic joining x and a point $y \in B_\delta \setminus \{x\}$.

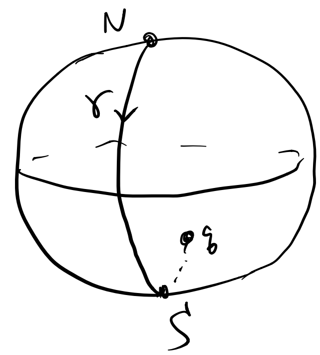
Then $L(\gamma) \leq L(\sigma)$ for all piecewise smooth curve σ on M (not necessary within B_δ) joining x to y.

Equality holds $\Leftrightarrow$ σ = monotonic reparametrization of γ.



Cor: Let $\gamma : [0,c] \to M$ be a arc-length parametrized piecewise smooth curve such that $L(\gamma) \leq L(\sigma)$ for all piecewise smooth curve σ joining $\gamma(0)$ and $\gamma(c)$. Then γ is a geodesic.

Caution: The converse of the Cor. is not true in general:



γ = geodesic, but not length minimizing.

<u>Def</u>: A geodesic $\gamma : [0, C] \to M$ is called a <u>minimizing geodesic</u> if $L(\gamma) \le L(\sigma)$ $\forall\, \sigma$ joining $\gamma(0)$ & $\gamma(C)$.

<u>Pf of Cor</u> (by assuming the Thm.)

Let $x = \gamma(0)$. Choose B_δ as in the thm.

Let $t_1 = \min\{t : \gamma(t) \in \partial B_\delta\}$

(If t_1 doesn't exist, then we are done.)

If $\gamma|_{[0,t_1]}$ is not geodesic, then by the theorem

$$L(\gamma|_{[0,t_1]}) > L(\gamma_1)$$

where γ_1 = radial geodesic joining $x = \gamma(0)$ to $\gamma(t)$ in B_δ.

$$\Rightarrow \quad L(\gamma_1 \cup \gamma|_{[t_1, C]}) < L(\gamma)$$ which is a contradiction.

Hence $\gamma|_{[0,t_1]}$ is a geodesic.

Continuing this argument $\Rightarrow$ $\gamma|_{[0,c]}$ is a geodesic. $\divideontimes$

Pf of the Thm

As in the proof of the Gauss Lemma, we can find bases $\{R, Y_1, \cdots, Y_{n-1}\}$ of T_zM for $z \in B_\delta \setminus \{x\}$,

such that R = tangential to the radial geodesic (and $|R| = 1$)

$Y_1, \cdots, Y_{n-1}$ = tangential to the geodesic sphere.

WLOG, we may assume $\sigma \subset B_\delta$.

Then for any such $\sigma : [0, 1] \to B_\delta$ s.t.

$$\sigma(0) = x, \quad \sigma(1) = y,$$

we have $\forall\, t \in [0, 1]$

$$\sigma'(t) = f(t)\, R(\sigma(t)) + T(t)$$

for some function $f(t)$,

where $T(t)$ = a linear combination of Y_i's.

Let $v \in B(\delta)$ be the unique vector st.

$$\exp_x(v) = y.$$

Then $\xi = \exp_x^{-1} \circ \sigma$ is a curve in $B(\delta) \subset T_xM$ joining 0 and v.

Since $(d\exp_x^{-1})(R) = \mathcal{R}$ $\left(= \text{radial unit vecta field defined above}\right)$

$$(d\exp_x^{-1})(Y_i) = \text{tangential to } S^{n-1}_{|\xi(t)|} \subset T_xM,$$

we see that

$$(d\exp_x^{-1})(\langle \sigma', R\rangle R) = f\,\mathcal{R} \quad (\text{by Gauss lemma})$$

is the radial projection of the tangent vecta ξ'

$$\Rightarrow \quad |v| = |\xi(1)| - |\xi(0)| = \int_0^1 f(t)\,dt$$

Hence $\quad L(\gamma) = |v| = \int_0^1 f(t)\,dt$

where γ is the radial geodesic joining x to $y = \exp_x v$.

Gauss Lemma again $\Rightarrow$

$$|\sigma'(t)|^2 = f^2(t)\,|R(\sigma(t))|^2 + |T(t)|^2 = f(t)^2 + |T(t)|^2$$

$$\Rightarrow \quad L(\sigma) = \int_0^1 |\sigma'| = \int_0^1 \sqrt{f^2 + |T|^2} \geq \int_0^1 f(t)\,dt = L(\gamma).$$

Finally, if $L(\sigma) = L(\gamma)$, then $T(t) = 0$ & $f > 0$.

$\Rightarrow \quad \sigma'(t) = f(t) R(\sigma(t))$ with $f > 0$

$\Rightarrow \quad \sigma =$ monotonic reparametrization of γ. ✱

§4.3 Completeness, metric structure

$(M, g) =$ Riemannian manifold (connected)

Def: $d : M \times M \to [0, \infty)$ defined by

$$d(x,y) = \inf_{\gamma} L(\gamma),$$

where "inf" is taken over all piecewise smooth curves γ joining x and y, is called the distance (a metric) of (M, g).

Thm: (M, d) is a metric space, i.e. d satisfies

(1) $d(x,y) \geq 0$; "=" iff $x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Pf: All are easy (Ex.) and we prove only

"$d(x,y)=0 \Rightarrow x=y$".

Suppose $x \neq y$.

If $y \in B_\delta$, where δ is given as in the "Thm" in §4.2, then

$$d(x,y) = L(\gamma), \text{ where } \gamma = \text{radial geodesic from } x \text{ to } y.$$

$$\Rightarrow \quad d(x,y) > 0.$$

Continuity argument $\Rightarrow$ $d(x,y) = \delta > 0$ if $y \in \partial B_\delta$

Hence if $y \notin B_\delta$ and σ = curve joining x to y.

Choose the 1st point y_1 of σ on ∂B_δ and conclude that

$$L(\sigma) \geq L\left(\sigma\big|_{(\text{from } x \text{ to } y_1)}\right) \geq \delta > 0.$$

Taking "inf" $\Rightarrow$ $d(x,y) \geq \delta > 0$. #

In fact, we have a stronger theorem

<u>Thm</u>: The topology of (M,d) is the same as the original topology of M.

(Pf: Ex, or pg 61-62 of H. Wu, or doCarmo.)

Def: A Riemannian manifold (M, g) is said to be complete if the associated metric space (M, d) is complete.

egs: $(\mathbb{R}^n$, standard metric$)$, $(S^n$, standard metric$)$ are complete.

Hopf-Rinow Theorem: The following statements are equivalent on a Riemannian manifold (M, g):

(1) M is complete,

(2) $\forall x \in M$, $\exp_x$ defined on the whole T_xM,

(3) $\exists x \in M$, $\exp_x$ defined on the whole T_xM,

(4) bounded closed subsets of M are compact.

Cor 1 of Hopf-Rinow Thm

If (M, g) is complete, then $\forall x \neq y \in M$, $\exists$ a minimizing geodesic γ joining x and y.

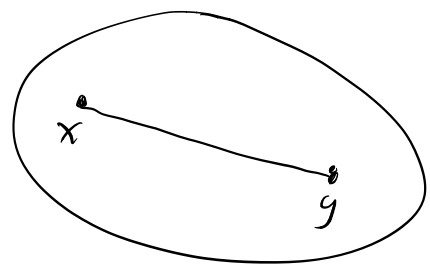
(Recall: all manifolds are connected in this course!)

Cor 2: If (M, g) is complete, then $\forall x \in M$,

$$\exp_x : T_xM \to M \text{ is surjective.}$$

Notes: • The converse of Cor 1 of Hopf-Rinow Thm is not true in general:

eg. convex subset in $\mathbb{R}^n$



• A general complete metric space may not have Heine-Borel property:

eg: $S = \{a_1, a_2, \cdots\}$ countable infinite set of distinct elements.

Define discrete metric d on S by

$$d(a_i, a_j) = 1 - \delta_{ij}$$

Then (S, d) is a complete metric space which is bounded

$\Rightarrow$ S is a closed and bounded set, but not compact.

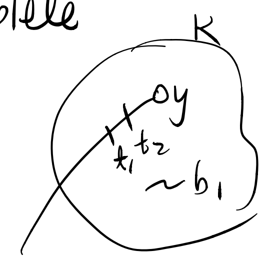
Pf of Hopf-Rinow Thm

(1) $\Rightarrow$ (2) Let $\gamma : [0, \delta) \to M$ be a geodesic

$\gamma(t) = \exp_x(tv)$ for some $v \in T_xM$. ($|v|=1$)

Suppose that $I = (a_1, b_1)$ is the maximal possible interval containing $[0, \delta)$ such that $\gamma(t)$ is defined. Suppose $b_1 < +\infty$. Then M complete

$\Rightarrow \exists\, y \in M$ s.t. $\lim_{t \to b_1} \gamma(t) = y$.



Let K = cpt nbd. of y

ODE theory $\Rightarrow \exists\, \varepsilon > 0$ indep. of t_0 s.t.

" If $d(\gamma(t_0), y) < \frac{\varepsilon}{2}$ ($\Rightarrow \gamma(t_0) \in K$),

then $\exists$ normalized geodesic $\zeta : [0, \varepsilon] \to M$

s.t. $\zeta(0) = \gamma(t_0)$ & $\zeta'(0) = \gamma'(t_0)$ "

$\Rightarrow$ joining ζ to γ gives an extension of γ beyond b_1.

Hence $b_1 = +\infty$.

Similar argument $\Rightarrow a_1 = -\infty$

$\therefore$ $\exp_x(tv)$ defined $\forall t \in (-\infty, \infty)$.

Since v (with $|v|=1$) is arbitrary, $\exp_x$ defined on whole T_xM.

(2) $\Rightarrow$ (3) trivial.

(4) $\Rightarrow$ (1) is standard for metric space.

To prove (3) $\Rightarrow$ (4), we claim that

(3) $\Rightarrow$ (5), where

(5) Assume $x \in M$ as in (3), then $\forall y \in M$, $\exists$ a minimizing geodesic joining x to y.

Pf of claim (5) (i.e. (3) $\Rightarrow$ (5))

Let $\overline{B}(r) = \{ y \in M : d(x,y) \le r \}$

$\Sigma(r) = \left\{ y \in \overline{B}(r) : y \text{ is joined to } x \text{ by a min. geodesic.} \right\}$

Then we need to show that

$$\overline{B}(r) = \Sigma(r), \quad \forall r \in [0, \infty).$$

Let $J = \{ r \in [0,\infty) : \overline{B}(r) = \Sigma(r) \}$.

(i) We have shown that

if $r<\delta$ where $\delta>0$ is given by the "Thm" in §4.2 (minimizing radial geodesic)

then $r \in \mathcal{J}$

$\Rightarrow \mathcal{J} \neq \emptyset$.

(ii) Since $\exp_x$ defined on whole $T_xM \cong \mathbb{R}^n$, continue dependence of $\exp_x(tv)$ on v

$\Rightarrow \mathcal{J}$ is closed.

(iii) To show $\mathcal{J}$ is open, we need the following fact (Ex, or see do Carmo)

(*) $\left[\begin{array}{l} \forall \text{ cpt } K \subset M, \exists \varepsilon>0 \text{ such that} \\ \forall y, z \in K \text{ with } d(y,z) \leq \varepsilon, \\ \text{then } \exists \text{ a minimizing geodesic joining } y \,\&\, z. \end{array}\right.$

Note: This is a stronger result than the Thm in §4.2 in which one of the points has to be the center.

<u>Pf of openess</u>: Let $r \in \mathcal{J}$, then $\overline{\mathcal{B}}(r) = \Sigma(r)$

$\therefore \quad \overline{\mathcal{B}}(r) \subset \exp(\overline{B(r)}) \Rightarrow \overline{\mathcal{B}}(r)$ is compact.

$\Rightarrow \partial\overline{\mathcal{B}}(r)$ is also compact.

Now $\forall\, z \in \partial\overline{B}(r)$, $\exists\, \varepsilon_1(z) > 0$ such that

$\exp_z : B(\varepsilon_1) \longrightarrow B_{\varepsilon_1}(z)$ is a diffeomaphism.

By compactness of $\overline{B}(r)$, $\exists$ finitely many z_i

such that $\{ B_{\frac{1}{2}\varepsilon_1(z_i)}(z_i) \}$ covers $\partial\overline{B}(r)$. This implies

$K = \overline{B}(r) \cup \left(\bigcup_i \overline{B_{\frac{3}{4}\varepsilon_1(z_i)}(z_i)} \right)$ is a compact set

that contains $\overline{B}(r+\varepsilon_2)$ for some $0 < \varepsilon_2 < \frac{1}{2}\varepsilon_1$.

Applying $(*)$, $\exists\, \varepsilon > 0$ with property in $(*)$.

Let $\varepsilon' \in (0, \min\{\varepsilon_2, \varepsilon\})$ and consider

$y \in \overline{B}(r+\varepsilon')$

If $y \in \overline{B}(r)$, then $y \in \Sigma(r)$ (since $r \in I$)

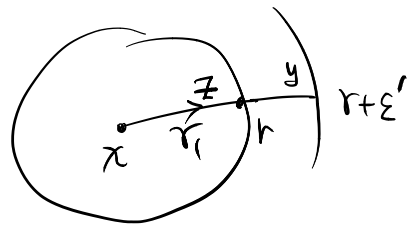
$$\Sigma(r) \subset \Sigma(r+\varepsilon')$$

If $y \in \overline{B}(r+\varepsilon') \setminus \overline{B}(r)$,

then $\exists\, z \in \partial\overline{B}(r)$ s.t.

$$d(x,y) = d(x,z) + d(z,y)$$

$\left(\begin{array}{l} \text{by using cptness of } \partial\overline{B}(r) \\ \text{\& definition of } d(x,y) \end{array} \right)$



Then $r \in \mathcal{I} \Rightarrow$

$\exists$ minimizing geodesic γ_1 joining x & z.

On the other hand,

$$d(z,y) = d(x,y) - d(x,z) \leq r + \varepsilon' - r = \varepsilon' < \varepsilon$$

$(*) \Rightarrow \exists$ minimizing geodesic γ_2 joining z & y.

Then connecting γ_1 & γ_2, we have a piecewise

smooth curve joining x & y with

$$\text{length} = d(x,z) + d(z,y) = d(x,y)$$

$\Rightarrow$ it must be a minimizing geodesic.

$\therefore\ y \in \Sigma(r+\varepsilon')$

Hence $\overline{B}(r+\varepsilon') \subset \Sigma(r+\varepsilon') \subset \overline{B}(r+\varepsilon')$.

$\therefore\ \mathcal{I}$ is open.

Hence (i), (ii) & (iii) $\Rightarrow \mathcal{I} = [0, \infty)$ $\therefore$ we've proved

that (3) $\Rightarrow$ (5).

<u>(3) $\Rightarrow$ (4)</u>

$\forall$ bounded & closed set K, $\exists A > 0$ s.t.

$d(x,k) \leq A$, $\forall\, k \in K$.

$\Rightarrow$ $K \subset \exp_x(\overline{B}(A))$ (by assumption (3))

$\Rightarrow$ K is cpt. (since $\overline{B}(A)$ is cpt.)

This completes the proof of Hopf–Rinow Thm. $\#$

Pf of Cor 1 : Hopf–Rinow $\Rightarrow$ (2) is true

($\Rightarrow$ (3) is true)

$\Rightarrow$ (5) is true $\forall\, x \in M$

$\Rightarrow$ Cor 1. is true. $\#$