Similarly for $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

$$\boxed{dx\wedge dy\wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\wedge dv\wedge dw} \qquad (\text{Ex!})$$

(using $dx = x_u du + x_v dv + x_w dw, \cdots$)

- "Oriented" change of variables formula
- "$dx\wedge dy$" oriented area element
- "$dx\wedge dy\wedge dz$" oriented volume element.

Exterior differentiation "d" on a form "ω".

0-form f	df (1-form)
1-form $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$	$d\omega = d\omega_1\wedge dx + d\omega_2\wedge dy + d\omega_3\wedge dz$ (2-form)
2-form $\xi = \xi_1 dy\wedge dz + \xi_2 dz\wedge dx + \xi_3 dx\wedge dy$	$d\xi = d\xi_1\wedge dy\wedge dz + d\xi_2\wedge dz\wedge dx + d\xi_3\wedge dx\wedge dy$ (3-form)
3-form $f\, dx\wedge dy\wedge dz$	$df\wedge dx\wedge dy\wedge dz = 0$ (4-form in $\mathbb{R}^3$)

e.g. ① $d(dx) = d(dy) = d(dz) = 0$. ($d^2x = d^2y = d^2z = 0$)

eg1 (in $\mathbb{R}^2$) $\quad \omega = M\,dx + N\,dy \quad (M = M(x,y),\ N = N(x,y))$

then
$$\begin{aligned} d\omega &= dM \wedge dx + dN \wedge dy \\ &= (M_x dx + M_y dy) \wedge dx + (N_x dx + N_y dy) \wedge dy \\ &= (N_x - M_y)\, dx \wedge dy \end{aligned}$$

(+ve) oriented area

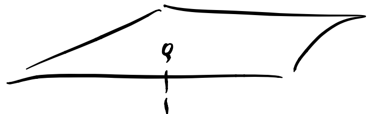
In this notation, Green's Thm $\displaystyle\oint_{C=\partial R} M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$,

can be written as
$$\boxed{\oint_{C=\partial R} \omega = \iint_R d\omega}$$

Remark: If we let $\vec{F} = M\hat{i} + N\hat{j} \leftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dA = (N_x - M_y)\underbrace{\hat{k}\cdot\hat{n}\,dA}_{dx \wedge dy} = d\omega$

($\hat{n} = \hat{k}$)

Hence, if we use  &

$\hat{n} = -\vec{k}$

$$\hat{k}\cdot\hat{n}\,dA = \begin{cases} dx \wedge dy & \text{if } \hat{n} = \vec{k} \\ dy \wedge dx & \text{if } \hat{n} = -\vec{k} \end{cases}$$

orientation of the "surface"

eg2: $\xi = \xi_1\, dy \wedge dz + \xi_2\, dz \wedge dx + \xi_3\, dx \wedge dy$

Then
$$\begin{aligned} d\xi &= d\xi_1 \wedge dy \wedge dz + d\xi_2 \wedge dz \wedge dx + d\xi_3 \wedge dx \wedge dy \\ &= \left(\frac{\partial \xi_1}{\partial x} dx + \cdots\right) \wedge dy \wedge dz \end{aligned}$$

$$+\left(\cdots+\frac{\partial\xi_2}{\partial y}dy+\cdots\right)\wedge dz\wedge dx$$

$$+\left(\cdots+\frac{\partial\xi_3}{\partial z}dz\right)\wedge dx\wedge dy$$

$$=\left(\frac{\partial\xi_1}{\partial x}+\frac{\partial\xi_2}{\partial y}+\frac{\partial\xi_3}{\partial z}\right)dx\wedge dy\wedge dz$$

$$=\operatorname{div}\vec{F}\,dx\wedge dy\wedge dz$$

where $\vec{F}=\xi_1\hat{i}+\xi_2\hat{j}+\xi_3\hat{k}$

Hence the divergence theorem can be written as:

$$\iiint_D d\xi=\iiint_D\left(\frac{\partial\xi_1}{\partial x}+\frac{\partial\xi_2}{\partial y}+\frac{\partial\xi_3}{\partial z}\right)\underbrace{dx\wedge dy\wedge dz}$$

(+ve) oriented volume

$$=\iiint_D\operatorname{div}\vec{F}\,dV=\iint_{S=\partial D}\vec{F}\cdot\hat{n}\,d\sigma$$

outward

To see the relation between $\vec{F}\cdot\hat{n}\,d\sigma$ and ξ,

we parametrize S:

$$\vec{r}(u,v)=x(u,v)\hat{i}+y(u,v)\hat{j}+z(u,v)\hat{k}$$

$$\Rightarrow\begin{cases}\vec{r}_u=x_u\hat{i}+y_u\hat{j}+z_u\hat{k}\\ \vec{r}_v=x_v\hat{i}+y_v\hat{j}+z_v\hat{k}\end{cases}$$

$$\Rightarrow\quad\vec{r}_u\times\vec{r}_v=\begin{vmatrix}y_u & y_v\\ z_u & z_v\end{vmatrix}\hat{i}+\begin{vmatrix}z_u & z_v\\ x_u & x_v\end{vmatrix}\hat{j}+\begin{vmatrix}x_u & x_v\\ y_u & y_v\end{vmatrix}\hat{k}$$

If $\vec{r}_u\times\vec{r}_v$ is <u>outward</u>, then

$$\hat{n}=\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\quad\text{and}\quad d\sigma=|\vec{r}_u\times\vec{r}_v|\,du\,dv$$

$$=|\vec{r}_u\times\vec{r}_v|\,du\wedge dv$$

(correct orientation)

then $\vec{F}\cdot\hat{n}\,d\sigma = \vec{F}\cdot\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,du\wedge dv$

$$= \left(\zeta_1\frac{\partial(y,z)}{\partial(u,v)} + \zeta_2\frac{\partial(z,x)}{\partial(u,v)} + \zeta_3\frac{\partial(x,y)}{\partial(u,v)}\right)du\wedge dv$$

$$= \zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy$$

$$= \zeta$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{(u,v)} \zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy$$

$$= \iint_{S=\partial D} \zeta$$

Hence divergence thm is

$$\boxed{\iiint_D d\zeta = \iint_{S=\partial D}\zeta}$$

ζ = 2-form

eg3 <u>Stokes' Thm</u>

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \Leftrightarrow \omega = M\,dx + N\,dy + L\,dz$$

Then $d\omega = (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx$ (Ex!)
$\quad + (N_x - M_y)\,dx\wedge dy$

$= (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$ (Ex!)

Stokes Thm becomes

$\oint_{C=\partial S}\vec{F}\cdot d\vec{r}$

$$\boxed{\oint_{C=\partial S}\omega = \iint_S d\omega}$$

$\iint_S(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$

Generalization to manifold of n-dimension with boundary (Skipped)

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with induced orientation)
- ω = $(n-1)$-form on M (smooth)

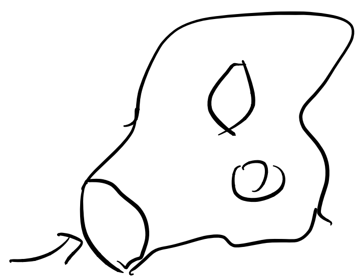
Then

$$\boxed{\underbrace{\int_M d\omega}_{\text{n-dim'l integral}} = \underbrace{\int_{\partial M} \omega}_{\text{(n-1)-dim'l integral}}}$$

Note: ∂M is always closed, i.e. no boundary.

$$\therefore \boxed{\partial(\partial M) = \partial^2 M = 0}$$

boundary has no boundary



∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η, then

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form

Ex: Verify this for 0-form and 1-form in $\mathbb{R}^3$ and observes that these are just

$$\begin{cases} \vec{\nabla} \times \vec{\nabla} f = 0 & (d^2 f = 0) \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 & (d^2\omega = 0) \end{cases}$$

eg: Let $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

Check: $d\omega = 0$

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$

(Since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2 \setminus \{(0,0)\}$)

Hence $\quad d\omega = 0 \not\Rightarrow \omega = d\eta$ in general

$(\Leftarrow)$

yes

<u>Note</u>: <u>Thm 10</u> can be written as:

$\Omega \subset \mathbb{R}^2$ simply-connected, then

$d\omega = 0 \Leftrightarrow \omega = df$ for some smooth function f on Ω.