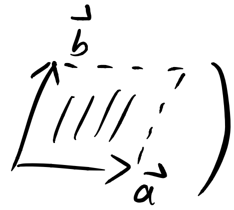
Surface Area

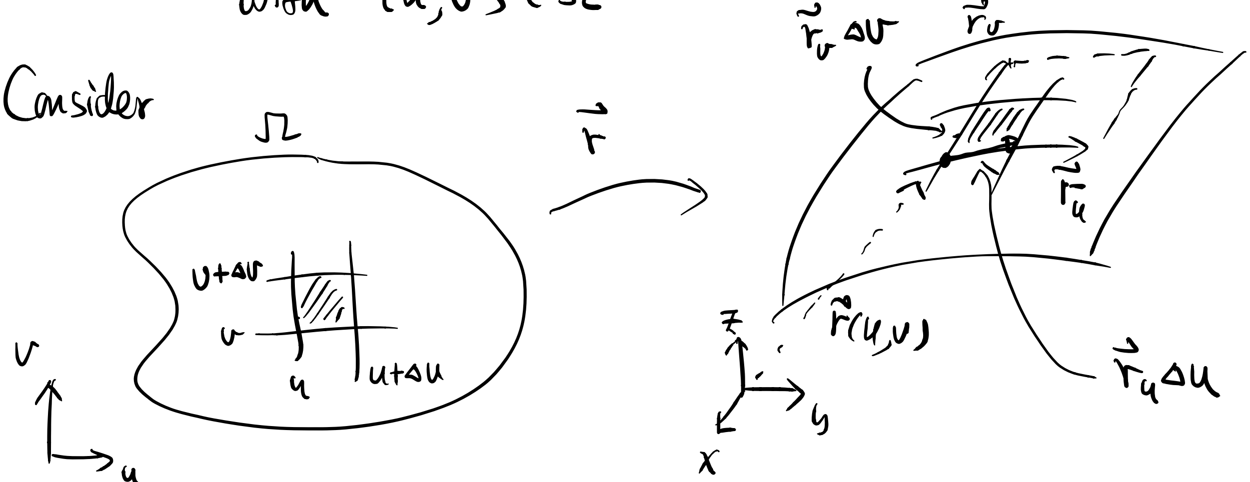
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

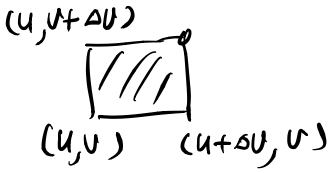
$$|\vec{a} \times \vec{b}| = \text{Area}\Big($$  $$\Big)$$

Let $\vec{r}(u,v)$ be a parametrization of a surface S

with $(u,v) \in \Omega$

Consider



$\Rightarrow$ "Area" on the surface corresponding to 

$$\text{is approx.} = |(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)|$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Hence "Area element" of S, denoted $d\sigma$,

is given by

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$\boxed{d\sigma = |\vec{r}_u \times \vec{r}_v| \, dA}$$

area element in the (u,v)-space.

Therefore, we make the following

Def 15: Let $S \subset \mathbb{R}^3$ be a smooth parametric surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$. Then

$$\text{Area}(S) \overset{\text{def}}{=} \iint_{\Omega} |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

(i.e. $\text{Area}(S) = \iint_{\Omega} d\sigma$)

eg52 Surface area of torus given by ($R > a > 0$ are constants)

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta & 0 \le \alpha \le 2\pi \\ y = (R + a\cos\alpha)\sin\theta & 0 \le \theta \le 2\pi \\ z = a\sin\alpha \end{cases}$$

i.e. $\vec{r}(\alpha, \theta) = (R + a\cos\alpha)\cos\theta \, \hat{i} + (R + a\cos\alpha)\sin\theta \, \hat{j} + a\sin\alpha \, \hat{k}$

$$\Rightarrow \begin{cases} \frac{\partial \vec{r}}{\partial \alpha} = -a\sin\alpha\cos\theta \, \hat{i} - a\sin\alpha\sin\theta \, \hat{j} + a\cos\alpha \, \hat{k} \\ \frac{\partial \vec{r}}{\partial \theta} = -(R + a\cos\alpha)\sin\theta \, \hat{i} + (R + a\cos\alpha)\cos\theta \, \hat{j} \end{cases}$$

&

$$\frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\alpha\cos\theta & -a\sin\alpha\sin\theta & a\cos\alpha \\ -(R+a\cos\alpha)\sin\theta & (R+a\cos\alpha)\cos\theta & 0 \end{vmatrix}$$

(check)

$$= -a(R + a\cos\alpha)\cos\theta\cos\alpha \, \hat{i} - a(R + a\cos\alpha)\sin\theta\cos\alpha \, \hat{j}$$
$$- a(R + a\cos\alpha)\sin\alpha \, \hat{k}$$

$$\left|\frac{\partial \vec{r}}{\partial \alpha}\times\frac{\partial \vec{r}}{\partial \theta}\right| = a(R+a\cos\alpha)\left[\cos^2\theta\cos^2\alpha+\sin^2\theta\cos^2\alpha+\sin^2\alpha\right]^{1/2}$$

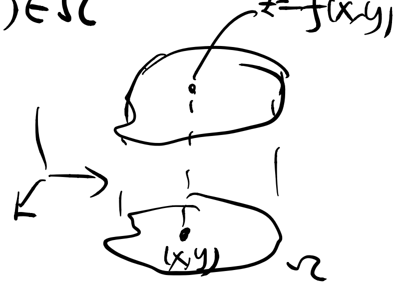
$$= a(R+a\cos\alpha)$$

Hence $\text{Area(Torus)} = \iint_{\Omega}\left|\frac{\partial \vec{r}}{\partial \alpha}\times\frac{\partial \vec{r}}{\partial \theta}\right| dA$

$$= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos\alpha)\,d\alpha\,d\theta$$

(check) $= 4\pi^2 Ra$ ##

<u>Surface area of a graph</u> $z=f(x,y)$, $(x,y)\in\Omega$



Choose the following parametrization of the graph

$$\vec{r}(x,y) = x\hat{i}+y\hat{j}+f(x,y)\hat{k}$$

$$\Rightarrow \begin{cases}\vec{r}_x = \hat{i}+f_x\hat{k}\\ \vec{r}_y = \hat{j}+f_y\hat{k}\end{cases}$$

$$\Rightarrow \quad \vec{r}_x\times\vec{r}_y = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix} = -f_x\hat{i}-f_y\hat{j}+\hat{k}$$

$$\Rightarrow \quad |\vec{r}_x\times\vec{r}_y| = \sqrt{f_x^2+f_y^2+1}$$

$$= \sqrt{1+|\vec{\nabla}f|^2} \geq 1$$ (nonzero, hence "smooth") if f is C^1

Thm 11: The surface area of a C^1 graph S given by

$$z = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

is

$$\text{Area}(S) = \iint_\Omega \sqrt{1+f_x^2+f_y^2}\, dA = \iint_\Omega \sqrt{1+|\vec{\nabla} f|^2}\, dA$$

(Similarly for $x = f(y,z)$ or $y = f(x,z)$)

Implicit Surface (level surface)

Suppose S is given by $F(x,y,z) = c$,

i.e. $S = F^{-1}(c)$

(Note: F is a function of 3-variables, not vector field)

eg 53: $F(x,y,z) = x^2+y^2+z^2$

Is $F^{-1}(0)$ a surface ?

No, since $F^{-1}(0) = \{(0,0,0)\}$, not a surface !

Remark: If $\vec{\nabla} F \neq 0$ at a point, then IFT implies that $S = F^{-1}(c)$ is a "surface" (c = value of F at that point) near that point (in fact, a graph!)

eg 53 (cont'd) $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore \quad \vec{\nabla} F = 0 \Leftrightarrow (x,y,z) = (0,0,0)$

Hence if $c > 0$, then $\forall (x,y,z) \in F^{-1}(c)$, we have

$\vec{\nabla} F(x,y,z) \neq 0$ $\left(\text{since } x^2+y^2+z^2 = c > 0 \Rightarrow (x,y,z) \neq (0,0,0) \right)$

$\Rightarrow$ $S = F^{-1}(c)$ $(\forall c>0)$ is a surface.

<u>Terminology</u> : $S = F^{-1}(c)$ is said to be <u>smooth</u> if (1) F is C^1 on S, and (2) $\vec{\nabla} F \neq 0$ on S.

(i.e. $S = F^{-1}(c)$ is a <u>smooth</u> level surface.)

How to compute surface area for a smooth level surface $S = F^{-1}(c)$?

By $\vec{\nabla} F \neq 0$, at least one of the partial derivatives F_x, F_y & F_z is nonzero. Let assume $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar.)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{ F(x,y,z) = c \}$

can be written (locally) as a graph

$$z = f(x,y) \quad \text{(near the point)}$$

i.e. $$F(x,y,f(x,y)) = c \quad \text{(near the point)}$$

Then chain rule $\Rightarrow$

$$\begin{cases} f_x = -\dfrac{F_x}{F_z} \\ f_y = -\dfrac{F_y}{F_z} \end{cases}$$

Hence $$\text{Area}(S) = \iint_{\Omega} \sqrt{1+f_x^2+f_y^2}\, dA$$

$$= \iint_{\Omega} \sqrt{1+\frac{F_x^2}{F_z^2}+\frac{F_y^2}{F_z^2}}\, dxdy$$

$$= \iint_{\Omega} \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}\, dx\,dy$$

Thm 12 If $S = F^{-1}(c)$ is a smooth level surface such that $F_z \neq 0$ (and can be represented by an implicit function on a domain Ω).

Then

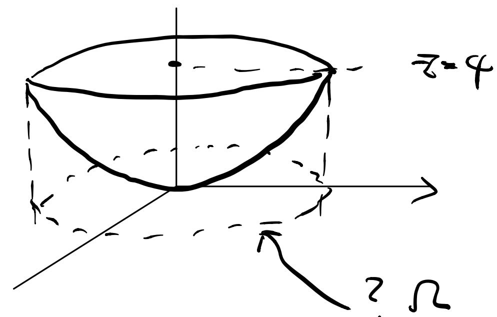
$$\text{Area}(S) = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$

(Similarly for the cases that $F_x \neq 0$ or $F_y \neq 0$)

eg 54: Find surface area of the paraboloid

$$x^2+y^2-z=0 \quad \text{below } z=4.$$

(This is in fact a graph. But we do it using level surface.)



Soln: Let $F(x,y,z) = x^2+y^2-z$.

For $z=4$, $x^2+y^2-z=0 \Rightarrow x^2+y^2=4$

$\Rightarrow$ projected region $\Omega = \{(x,y) : x^2+y^2 \leq 4\}$

Check: $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\Rightarrow \quad F_z = -1 \neq 0, \quad \forall (x,y) \in \Omega$

$$\Rightarrow \quad \text{Surface Area} = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA$$

$$= \iint_{\{x^2+y^2\le 4\}} \frac{\sqrt{4x^2+4y^2+1}}{|-1|}\, dA$$

$$= \iint_{\{x^2+y^2\le 4\}} \sqrt{4(x^2+y^2)+1}\, dA$$

(using polar coordinates)

$$\overset{\text{check}}{=} \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right] \quad \#$$

Def 16 Surface Integral (of a function)

Suppose $G: S \to \mathbb{R}$ is a continuous function on a surface S, parametrized by $\vec{r}(u,v)$, $(u,v)\in R$. Then the integral of G on S is

$$\iint_S G\, d\sigma \overset{\text{def}}{=} \iint_R G(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

element area of the parameter space $dA = du\,dv$

Note: In the cases of graph or level surface, we have

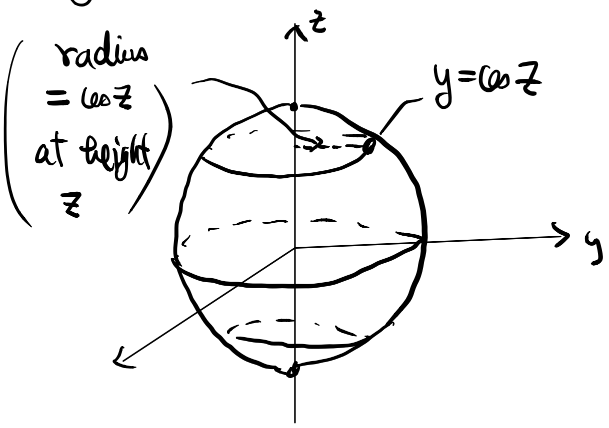
(i) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,f(x,y))\sqrt{1+|\nabla f|^2}\, dx\,dy$$

(for $z = f(x,y)$)

(ii) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,z)\, \frac{|\nabla F|}{|F_z|}\, dx\,dy$$

(for $F(x,y,z) = c$, $F_z \neq 0$)

eg56 (a surface of revolution of the curve $y=\cos z$)



$\left(-\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Let $G(x,y,z)=\sqrt{1-x^2-y^2}$ be a function on S.

Find $\iint_S G\, d\sigma$

Soln: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta \\ y = \cos z \sin\theta \, , \\ z = z \end{cases} \qquad \begin{matrix} -\pi \le \theta \le \pi \\ -\frac{\pi}{2} \le z \le \frac{\pi}{2} \end{matrix}$$

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\, \hat{i} + \cos z \sin\theta\, \hat{j} + z\, \hat{k}$

(Note: there is an exceptional set of "1-dim" which is not a parametric surface corresponds to $\theta = \pi$ or $-\pi$ & $z = -\frac{\pi}{2}$ or $\frac{\pi}{2}$)

$$\Rightarrow \begin{cases} \vec{r}_\theta = -\cos z \sin\theta\, \hat{i} + \cos z \cos\theta\, \hat{j} \\ \vec{r}_z = -\sin z \cos\theta\, \hat{i} - \sin z \cos\theta\, \hat{j} + \hat{k} \end{cases}$$

$\Rightarrow \vec{r}_\theta \times \vec{r}_z = \cos z \cos\theta\, \hat{i} + \cos z \sin\theta\, \hat{j} + \sin z \cos z\, \hat{k}$ (check!)

$\Rightarrow |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z\,(1+\sin^2 z)} = \cos z \sqrt{1+\sin^2 z}$

$\left(\cos z \ge 0 \text{ for } -\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Then $$\iint_S G\, d\sigma = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta,z))\, |\vec{r}_\theta \times \vec{r}_z|\, dz\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-x^2-y^2}\, |\vec{r}_\theta \times \vec{r}_z|\, dz\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 z} \cos z \sqrt{1+\sin^2 z} \, dz \, d\theta$$

$$\overset{\text{check}}{=} \cdots = 2\int_{-\pi}^{\pi} \int_{0}^{\frac{\pi}{2}} \sin z \cos z \sqrt{1+\sin^2 z} \, dz \, d\theta$$

$$= \frac{4\pi}{3}(2\sqrt{2}-1) \quad (\text{check!})$$

$\#$