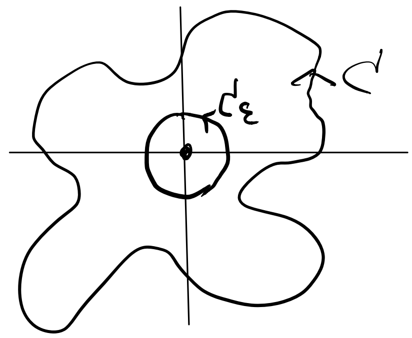
(a) Recall that even $\vec{\nabla}\times\vec{F}=0$ (check)

Green's Thm doesn't apply to get $\oint_C \vec{F}\cdot d\vec{r}=0$, since

C encloses the origin $(0,0)$ where $\vec{F}$ is not defined.



Choose $\varepsilon>0$ small enough, such that the circle C_ε of radius ε centered at $(0,0)$ is completely enclosed by C.

$\vec{F}$ is small in the region R between C and C_ε, Hence

the general form of Green's Thm gives

$$0=\iint_R \vec{\nabla}\times\vec{F}\cdot\vec{k}\ dA=\oint_C \vec{F}\cdot d\vec{r}-\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}$$

$$=\oint_{C_\varepsilon}\frac{-y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$$

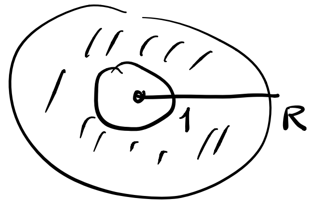
Parametrize C_ε by $\begin{cases} x=\varepsilon\cos\theta \\ y=\varepsilon\sin\theta \end{cases}$, $0\le\theta\le 2\pi$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left[-\frac{\varepsilon\sin\theta}{\varepsilon^2}(-\varepsilon\sin\theta)+\frac{\varepsilon\cos\theta}{\varepsilon^2}(\varepsilon\cos\theta)\right]d\theta$$

$$=\int_0^{2\pi}1\,d\theta=2\pi$$

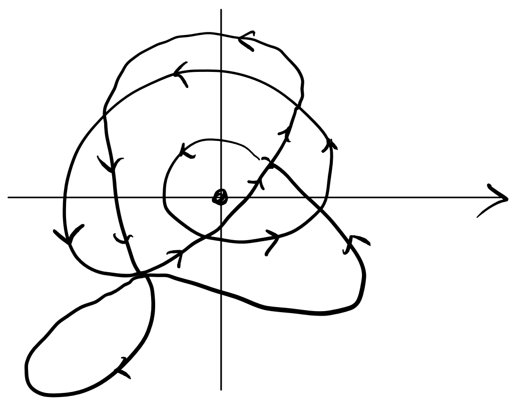
(In fact, we've proved that $\oint_{C_R}\vec{F}\cdot d\vec{r}=2\pi$, $\forall$ any radius $R>0$,

which can also proved by consider the domain between

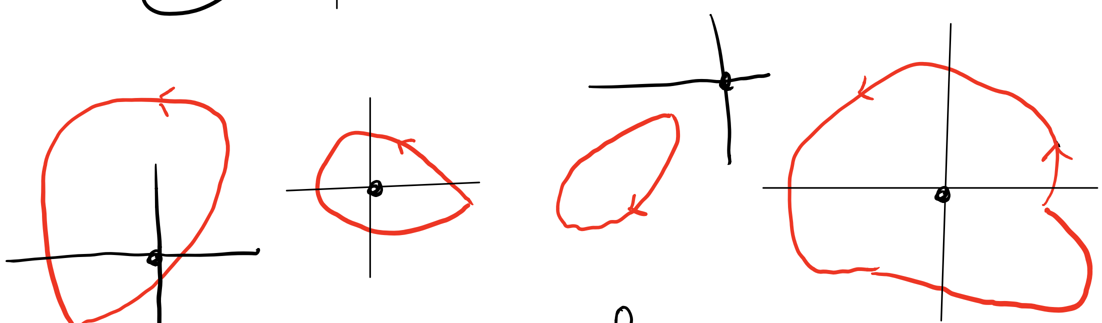
C_1 & C_R, 

and apply general form of Green's Thm to get $\oint_{C_R} \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$)

(b)



Recompose the curves into the following parts:



$2\pi \qquad 2\pi \qquad 0 \qquad 2\pi$

Hence $\oint_C \vec{F}\cdot d\vec{r} = 2\pi + 2\pi + 0 + 2\pi = 6\pi$

Surface Area & Integral

Def 14 Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a mapping of 2 variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

And it is called smooth if

(1) $\vec{r}$ is C^1 (i.e. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

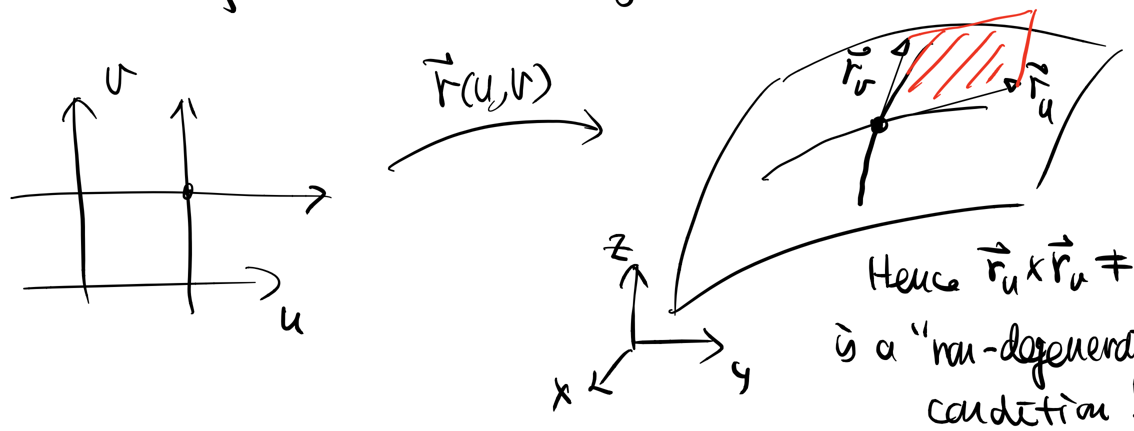
(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq 0}$, $\forall u, v$

where
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\ \qquad = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases}$$

Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linear independent

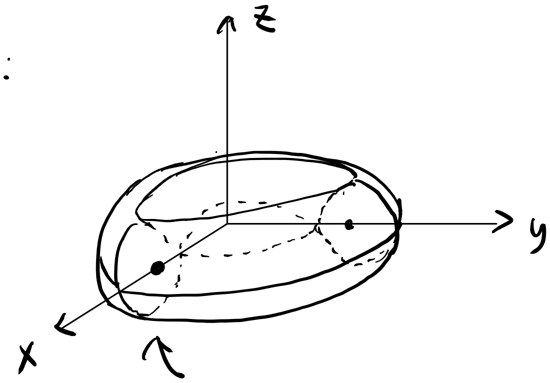
$\Rightarrow$ $\mathrm{span}\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dim'l plane

$\Rightarrow$ "surface" cannot be degenerated to a curve or point



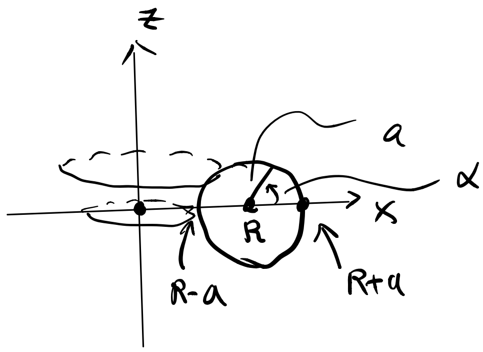
Hence $\vec{r}_u \times \vec{r}_v \neq 0$ is a "non-degenerate" condition!

eg51 :



a circle in xz-plane,

rotating this circle around the z-axis gives "torus"



For $y=0$ (i.e. xz-plane)

the circle can be parametrized by

$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases} \qquad 0 \le \alpha \le 2\pi$$

Revolving around the z-axis, we have

$$\begin{cases} x = (R+a\cos\alpha)\cos\theta & 0 \le \alpha \le 2\pi \\ y = (R+a\cos\alpha)\sin\theta & 0 \le \theta \le 2\pi \\ z = a\sin\alpha \end{cases}$$

is a parametrization of the torus

Note that this torus can also be described as

$$\left(\sqrt{x^2+y^2}-R\right)^2 + z^2 = a^2$$