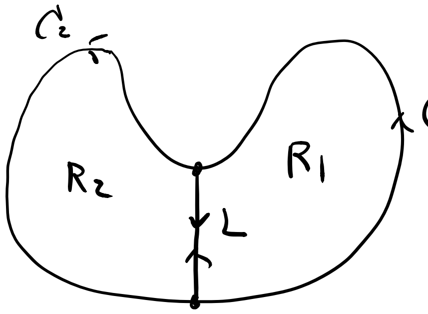
Proof of Green's Thm for

R = finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

[e.g.



R_1, R_2 = simple

but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L$ (L : $\downarrow$)

$\partial R_2 = C_2 - L$

with anti-clockwise orientation &

$\partial R = C_1 + C_2$.]

By assumption $R = \cup R_i$ finite union s.t.

R_i are simple and

$R_i \cap R_j$ = line segment of a common boundary portion, denote by L_{ij} $(i \neq j)$ (may be empty)

Then $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \sum_i \oint_{\partial R_i} M\,dx + N\,dy$$ (by the Green's Thm of simple region)

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at the end points)

Then $\partial R_i = C_i' + \sum_{j \neq i} L_{ij}$,

where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i' + \sum_{j (j \neq i)} L_{ij}} M\,dx + N\,dy$$

$$= \sum_i \int_{C_i'} M\,dx + N\,dy + \sum_i \int_{\sum_{j (j\neq i)} L_{ij}} M\,dx + N\,dy$$

Note that, as C_i' is not a common boundary of any other R_j,

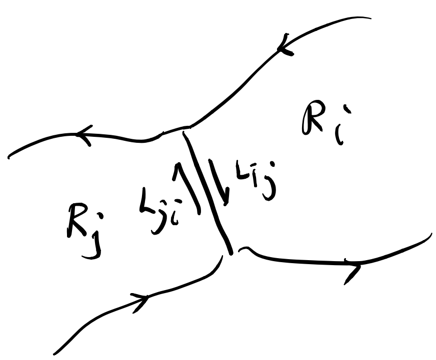
$$\sum_i C_i' = \partial R.$$

$$\therefore \quad \sum_i \int_{C_i'} M\,dx + N\,dy = \oint_{\partial R} M\,dx + N\,dy$$

Finally, we have

$$L_{ji} = -L_{ij}$$

as R_i & R_j are located on the two different sides of the common boundary.



$$\sum_i \int_{\sum_{\substack{j \\ (j\neq i)}} L_{ij}} Mdx+Ndy = \sum_i \sum_{\substack{j \\ (j\neq i)}} \int_{L_{ij}} Mdx+Ndy$$

$$= \sum_{\substack{i,j \\ i\neq j}} \int_{L_{ij}} Mdx+Ndy$$

$$= \sum_{i<j} \int_{L_{ij}} Mdx+Ndy + \sum_{\substack{\cancel{i}>\cancel{j} \\ j \quad i}} \int_{\substack{L_{\cancel{i}\cancel{j}} \\ j\,i}} Mdx+Ndy \quad \left(\begin{array}{l}\text{interchanging} \\ \text{the notation for} \\ \text{the dummy indexes}\end{array}\right)$$

$$= \sum_{i<j} \int_{L_{ij}} Mdx+Ndy + \sum_{i<j} \int_{L_{ji}} Mdx+Ndy$$

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx+Ndy + \int_{L_{ji}} Mdx+Ndy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx+Ndy + \int_{-L_{ij}} Mdx+Ndy \right) \quad (\text{since } L_{ji} = -L_{ij})$$

$$= 0$$

This 2nd case basically include almost all situations in the level of Advanced Calculus.

The proof of general case needs "analysis" and will be omitted here. $\#$

$\boxed{\text{Def 12: The } \underline{\text{divergence}} \text{ of } \vec{F} = M\hat{i} + N\hat{j} \text{ is defined to be } \quad \operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$

Note: $\operatorname{div}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$

$$= \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{n} \, ds$$

$\underset{=}{\text{(called)}}$ "flux density"

Notation: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (gradient)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} \right) f$$

It is convenient to denote

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}}$$

(∇ "nabla")

Then $\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} \right) \cdot (M\hat{i} + N\hat{j})$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}.$$

Hence we also write $\boxed{\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F}}$

Ref 13: Define $\operatorname{rot}\vec{F}$ to be

$$\operatorname{rot}\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad \text{for } \vec{F} = M\hat{i} + N\hat{j}$$

Note:

$$\operatorname{rot}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{T}\, ds$$

(called)
= circulation density

Using $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$, we can write

$$\boxed{\operatorname{rot}\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + 0\hat{k}$ (in $\mathbb{R}^3$)

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ (in $\mathbb{R}^3$)

$$\Rightarrow \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{k}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$\Rightarrow \quad \operatorname{rot}\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$ i.e. $\hat{k}$-component of "curl $\vec{F}$"

where $\boxed{\operatorname{curl}\vec{F} = \vec{\nabla} \times \vec{F}}$

Using the notations of $\begin{cases} \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} \\ \operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} \end{cases}$ and,

the Green's Thm can be written as

<u>Vector forms of Green's Thm</u>

<u>normal form</u>

$$\boxed{\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_D \operatorname{div} \vec{F} \, dA}$$

or

$$\boxed{\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_D \vec{\nabla} \cdot \vec{F} \, dA}$$

tangential form

$$\boxed{\oint_C \vec{F} \cdot \hat{T} \, ds = \iint_D \operatorname{curl} \vec{F} \cdot \hat{k} \, dA}$$

or

$$\boxed{\oint_C \vec{F} \cdot \hat{T} \, ds = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \, dA}$$

And Theorem 10 can be written as

<u>Thm 10'</u> Ω = simply-connected, $\vec{F} \in C^1$. Then

$$\vec{F} = \text{conservative} \Leftrightarrow \operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = 0$$

(Check: $n=3$ case)

<u>Note</u>: (i) $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$ defined only in $\mathbb{R}^3$ ($\supset \mathbb{R}^2$)

(ii) but $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ can be defined on $\mathbb{R}^n$ for any n.

In particular, in $\mathbb{R}^3$

Def 12' The <u>divergence</u> of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(M\hat{i} + N\hat{j} + L\hat{k}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts: (Ex!)

For C^2 function f and C^2 vector field $\vec{F}$:

(i) $\vec{\nabla}\times(\vec{\nabla}f) = 0$ (i.e. $\operatorname{curl}\vec{\nabla}f = 0$)

(ii) $\vec{F}$ conservative $\Rightarrow$ $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0$

(iii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ (i.e. $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$)

Remark: $\vec{\nabla}\cdot(\vec{\nabla}f) \neq 0$ in general, and it is called the <u>Laplacian</u> of f, and is denoted by

$$\vec{\nabla}^2 f = \vec{\nabla}\cdot(\vec{\nabla}f) = \operatorname{div}(\vec{\nabla}f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

[In graduate level, it will be denoted by $\Delta = \vec{\nabla}^2$ or $\Delta = -\vec{\nabla}^2$]

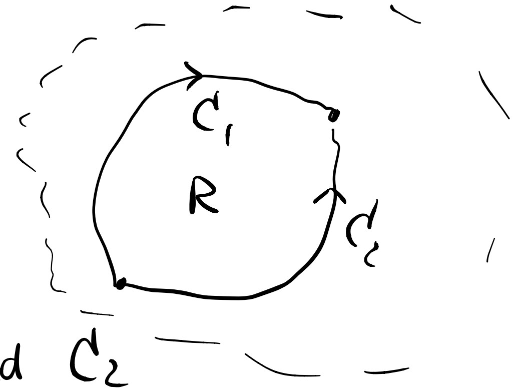
The "operator" $\vec{\nabla}^2$ is called the <u>Laplace operator</u> and the equation $\vec{\nabla}^2 f = 0$ is called the <u>Laplace equation</u>.

Solutions to Laplace equation are called <u>harmonic functions</u>.

Pf of Thm 10 ($n=2$)

We only need to show Ω simply-connected & $\vec{\nabla}\times\vec{F}=0$ $\left(\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\right)$ then $\vec{F}$ is conservative.

Suppose $C_1, C_2 \subseteq \Omega$ have the same starting point and end point.



Case 1: C_1, C_2 have no intersection

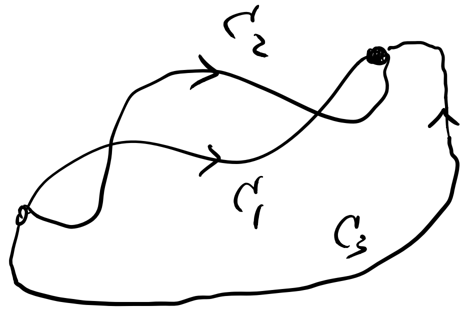
Then "Ω is simply-connected"

$\Rightarrow$ the region R enclosed by C_1 and C_2 lies completely inside Ω. Then by Green's Thm,

$$0=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA=\pm\left(\int_{C_1}-\int_{C_2}\right)(M\,dx+N\,dy)$$

$$\Rightarrow \quad \int_{C_1} M\,dx+N\,dy=\int_{C_2} M\,dx+N\,dy$$

Case 2: C_1, C_2 intersect



Pick another curve C_3 with the same starting point and end point, and do not intersect C_1 or C_2.

Then by Case 1, $\int_{C_1} M\,dx+N\,dy=\int_{C_3} M\,dx+N\,dy$

$$=\int_{C_2} M\,dx+N\,dy$$

$\therefore$ $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path and hence conservative #

In order to apply Green's Thm to more general situations, we need a more general form of Green's Thm:

Suppose we have a simple closed curve C in $\mathbb{R}^2$

Suppose $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.

(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented)

Let R be the region between C and $C_1, \cdots, C_n$.

Suppose $\vec{F} = M\hat{i} + N\hat{j}$ is defined on some open set containing R and is C^1. Then we have

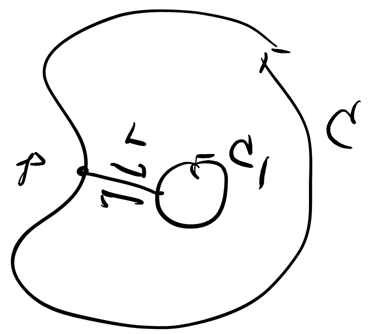
$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy}$$

(This is the tangential form. The normal form is similar.)

<u>Sketch of Proof</u>:

For simplicity, only one C_1 inside C.

We connect C & C_1 by an "arc" L and consider the "simple" closed



curve (starting from p):

$$C^* = C + L - C_1 - L$$

Then the region R enclosed between C & C_1 is the region enclosed by C^* except the arc L

Hence $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R\setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$$\underset{\text{Green's}}{=} \oint_{C^*} M\,dx + N\,dy$$

$$= \left(\oint_C + \int_L - \oint_{C_1} - \int_L\right)(M\,dx + N\,dy)$$

$$= \oint_C M\,dx + N\,dy - \oint_{C_1} M\,dx + N\,dy$$

#

eg 48: $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2 \setminus \{(0,0)\} = \Omega$

we've calculated $\oint_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$ for $C_1: x^2+y^2 = 1$ (anti-clockwise)

How about

(a) (b)