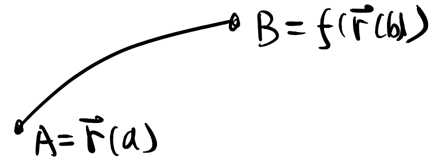
Pf of Fundamental Theorem of Line Integral:

Assume C is a curve parametrized by

$\vec{r}(t)$, $a \leq t \leq b$.



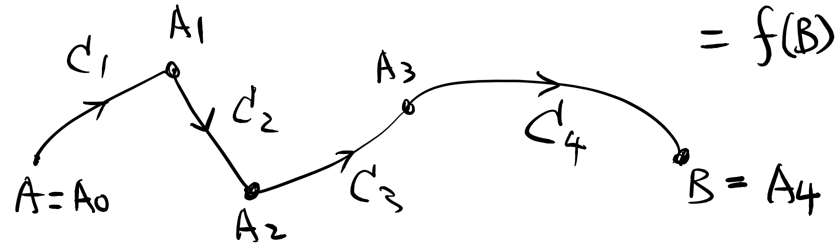
Then $\displaystyle\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \qquad \text{(chain rule)}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \qquad \text{Fundamental Thm of 1-variable.}$$

$$= f(B) - f(A)$$



$(k = 4)$

For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$

($= C_1 + C_2 + \cdots + C_k$ in order to indicate the orientation of C_i are correct wrt the orientation of C)

where C_i is smooth going from A_{i-1} to A_i

Then $\displaystyle\int_C \vec{F} \cdot \vec{T}\, ds = \sum_i \int_{C_i} \vec{F} \cdot \vec{T}\, ds$

$$= \sum_i \left[f(A_i) - f(A_{i-1}) \right]$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A) \qquad \left(\text{since } \begin{array}{l} A_0 = A \\ A_k = B \end{array} \right)$$

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Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open and connected.

$\vec{F}$ is a continuous vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f : \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_C \vec{F} \cdot d\vec{r} = 0$ along any closed curve C in Ω.

(c) $\vec{F}$ is conservative.

Pf "(a) $\Rightarrow$ (b)" If f is C^1 and $\vec{F} = \vec{\nabla} f$,

and $\vec{r} : [a,b] \to \Omega$ parametrizes C

C closed $\Rightarrow$ $\vec{r}(a) = \vec{r}(b) = A$

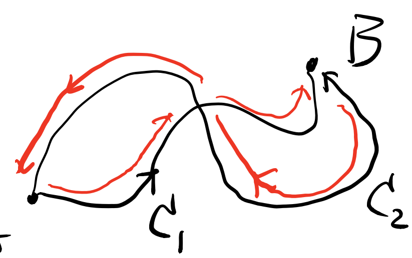
Fundamental Thm of Line Integral

$$\Rightarrow \oint_C \vec{F} \cdot \vec{T} \, ds = f(\vec{r}(b)) - f(\vec{r}(a)) = f(A) - f(A) = 0.$$

"(b) $\Rightarrow$ (c)" Suppose C_1, C_2 are oriented curves with starting point A and end points B.

Then $C_1 \cup (-C_2)$

$= C_1 - C_2$ (better notation)



is an (oriented) closed curve.

Then by (b) $0 = \oint_{C_1 - C_2} \vec{F} \cdot \vec{T} \, ds = \int_{C_1} \vec{F} \cdot \vec{T} \, ds + \int_{-C_2} \vec{F} \cdot \vec{T} \, ds$

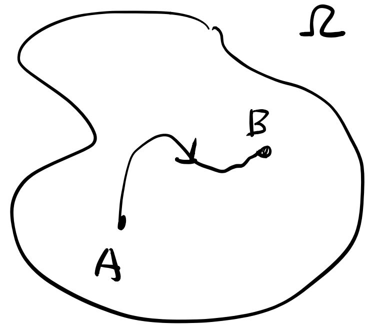
$$= \int_{C_1} \vec{F}\cdot\vec{T}\,ds - \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

$$\therefore \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds .$$

Since C_1, C_2 are arbitrary, $\vec{F}$ is conservative.

"(c)⇒(a)" Assume $n=2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ are conservative



Fix a point $A \in \Omega$.

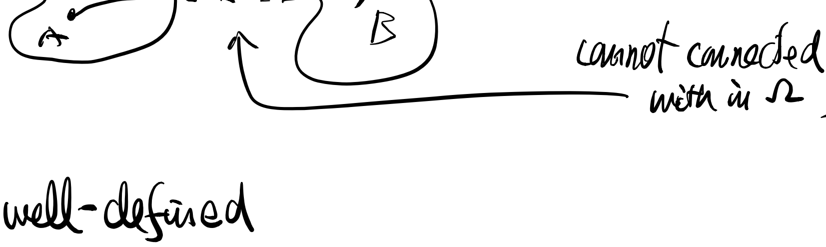
For any point $B \in \Omega$,

$$f(B) = \int_A^B \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} \quad \text{for } \underline{\text{any}}\ C \text{ from } A \text{ to } B.$$

(oriented curve)

(since $\vec{F}$ is conservative)

[We've also used the assumption that Ω is <u>connected</u>, otherwise there is no path from A to B if A, B belong to different connected components:



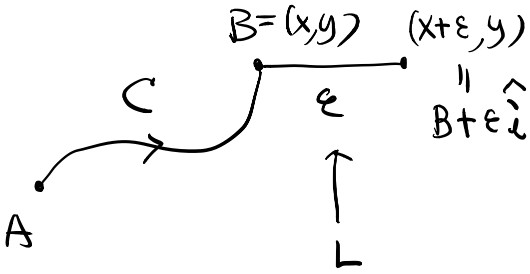
]

Hence $f(B)$ is well-defined

<u>Claim</u> $\vec{F} = \vec{\nabla} f$

<u>Pf of Claim</u>: $$\frac{\partial f}{\partial x}(B) = \lim_{\varepsilon\to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$$

Let C be an oriented curve from A to B.



Then $f(B+\varepsilon\hat{i})$

$$= \int_A^{B+\varepsilon\hat{i}} \vec{F}\cdot d\vec{r} = \int_{C+L} \vec{F}\cdot d\vec{r}$$

$L =$ horizontal line segment from B to $B+\varepsilon\hat{i}$ with $|\varepsilon|$ sufficient small ($s.t.$ $C+L \subset \Omega$)
open

$$= \int_C \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= \int_A^B \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= f(B) + \int_L \vec{F}\cdot d\vec{r}$$

$$\therefore \quad f(B+\varepsilon\hat{i}) - f(B) = \int_L \vec{F}\cdot d\vec{r}$$

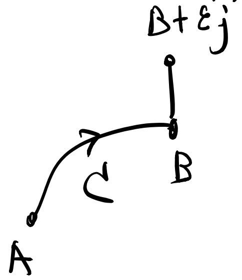
$$= \int_0^\varepsilon M(x+t,y)\,dt \quad \text{(check!)}$$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon M(x+t,y)\,dt$$

$$= M(x,y) \quad \text{(by MVF \& } M \text{ is continuous)}$$

$$\therefore \quad \frac{\partial f}{\partial x}(x,y) = M(x,y).$$

Similarly $\frac{\partial f}{\partial y}(x,y) = N(x,y)$ by considering



So $\vec{\nabla} f = \vec{F}$.

Since $\vec{F}$ is continuous, i.e. $M = \frac{\partial f}{\partial x}$ & $N = \frac{\partial f}{\partial y}$ are continuous,

f is C^1. ※

Remark: The function f in (a) of Thm 9 is called the potential function for $\vec{F}$. It is unique up an additive constant:

$$\vec{\nabla}(f+c)=\vec{F}, \quad \forall \text{ const. } c.$$