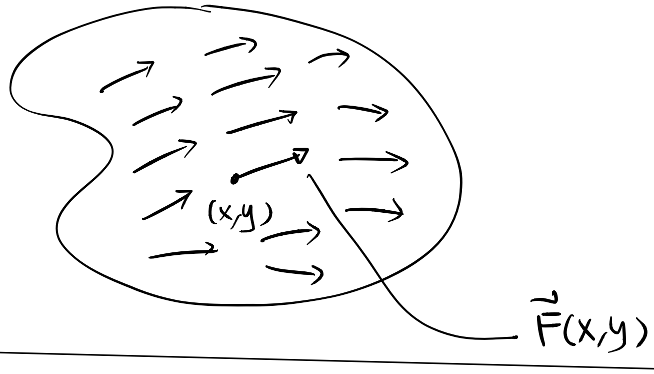
Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a vector field on D is a mapping $\vec{F} : D \longrightarrow \mathbb{R}^2$ or $\mathbb{R}^3$ respectively.



In component form:

$\mathbb{R}^2$: $\quad \vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$

$\mathbb{R}^3$: $\quad \vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + L(x,y,z)\,\hat{k}$

where M, N, L are functions on D called the components of $\vec{F}$.

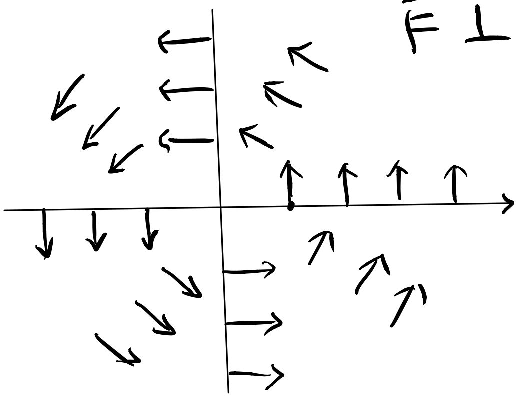
eg 35: $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ (in polar coordinates)

Properties of $\vec{F}$: $\quad |\vec{F}(x,y)| = 1$

$\vec{F} \perp \vec{r}(x,y) = x\hat{i} + y\hat{j}$

$= r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$



(Ex: sketch $\vec{r}(x,y) = x\hat{i} + y\hat{j}$)

eg36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j} = \vec{r}(x,y) = \vec{r}$$

(ii) $f(x,y,z) = x$

$$\vec{\nabla} f(x,y,z) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1,0,0) = \hat{i}$$

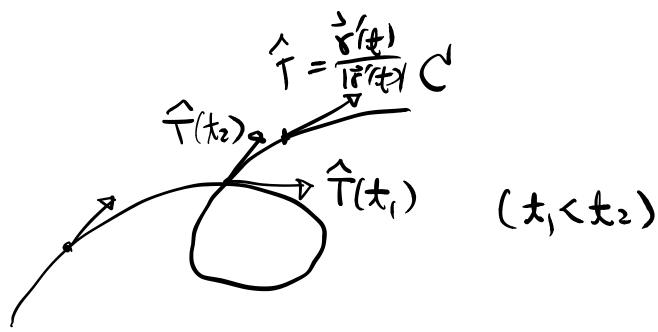
eg37 (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\begin{aligned} \vec{r} : [a,b] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (x(t), y(t)) = \vec{r}(t) \end{aligned}$$

Recall: $\hat{T}$ = unit tangent vector field along C

$$= \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$



Note that this vector field only defined on C, but not outside C

Remark for eg37:

If we use $ds = |\vec{r}'(t)|\,dt$, then

$$\hat{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds} \quad \text{(by chain rule)}$$

where "arc-length s" is defined (up to an additive constant) by

$$s(t) = \int_{t_0}^{t} |\vec{r}'(t)|\,dt .$$

A parametrization of a curve C by arc-length s is called

<u>arclength parametrization</u> :

$$\vec{r}(s) = \text{arc-length parametrization}$$

$$\Rightarrow \quad \left|\frac{d\vec{r}}{ds}(s)\right| = 1 .$$

<u>Def 11</u> A vector field is defined to be <u>continuous</u>/<u>differentiable</u>/C^k if the <u>component functions</u> are.

<u>eg 38</u> :

$\vec{F}(x,y) = \vec{r}(x,y) = x\hat{i} + y\hat{j}$ is C^∞,

$\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ is <u>not</u> continuous in $\mathbb{R}^2$

(but continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$)

<u>Line integral of vector field</u>

<u>Def 12</u> : Let C be a curve with <u>orientation</u> given by a parametrization $\vec{r}(t)$ with $\vec{r}\,'(t) \neq \vec{0}$, $\forall t$. Define the <u>line integral of a vector field</u> $\vec{F}$ along C to be

$$\int_C \vec{F} \cdot \hat{T} \, ds$$

where $\hat{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector field along C.

(i.e. C is oriented in the direction of $\vec{r}\,'(t)$ or $\hat{T}$ at every point)

Note: If $\vec{r}: [a,b] \to \mathbb{R}^n$ ($n=2$, or 3) then

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}\underbrace{|\vec{r}'(t)|\,dt}_{ds}$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\underbrace{\vec{r}'(t)\,dt}_{d\vec{r}}$$

$\therefore$ naturally, we write

$$\boxed{d\vec{r} = \hat{T}\,ds} \quad \text{and}$$

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}}$$

eg 38: $\vec{F}(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$, $0 \le t \le 1$

Then $$d\vec{r} = \left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$$

and $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \left(\sqrt{t}\,\hat{i} + t^2\cdot t\,\hat{j} - t^2\hat{k}\right)\cdot\left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$$

$$= \int_0^1 \left(2t\sqrt{t} + t^3 - \frac{t^{3/2}}{2}\right)dt = \frac{17}{20} \text{ (check!)}$$

Line Integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$C: \vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j}$ can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_a^b (Mg' + Nh')\,dt \quad \left(\begin{array}{l}\text{explicitly:}\\ M(g(t),h(t))\,g'(t) + N(g(t),h(t))\,h'(t)\end{array}\right)$$

Similarly for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ along

$$C : \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + f(t)\vec{k}$$

is

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_a^b (Mg' + Nh' + Lf')\,dt \qquad (\text{check!})$$

Note: usually, people write $\begin{cases} dx = g'(t)dt \\ dy = h'(t)dt \\ dz = f'(t)dt \end{cases}$

$$\Rightarrow \boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C M\,dx + N\,dy + L\,dz}$$

Similarly for $\mathbb{R}^2$: $\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C M\,dx + N\,dy}$

Another way to justify the notation:

$\vec{r} = (x, y, z)$ the position vector

$\Rightarrow \boxed{d\vec{r} = (dx, dy, dz)}$ (naturally)

Then $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (M, N, L)\cdot(dx, dy, dz)$$

$$= \int_C M\,dx + N\,dy + L\,dz.$$

eg 39: Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C: \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k} \quad (0 \le t \le 2\pi)$

$$= (\cos t, \sin t, t)$$

Soln: $d\vec{r} = (-\sin t, \cos t, 1)\,dt$

$$\Rightarrow I = \int_0^{2\pi} [-\sin t\,(-\sin t) + t(\cos t) + 2\cos t\,(1)]\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt = \pi \quad \#$$

Physics

(1) $\vec{F}$ = Force field

C = oriented curve,

then $\boxed{W = \int_C \vec{F} \cdot \hat{T}\, ds}$

is work done in moving an object along C

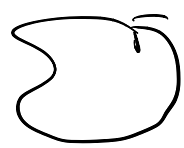
(2) $\vec{F}$ = velocity vector field of fluid

C = oriented curve

Then $\boxed{\text{Flow} = \int_C \vec{F} \cdot \hat{T}\, ds}$

Flow along the curve C

If C is a closed curve, the flow is also called the circulation



C

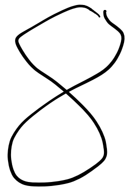
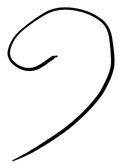
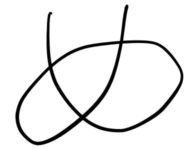
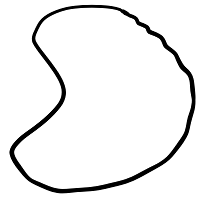
Def 13: A curve is said to be

(i) simple if it does not intersect with itself except possibly at end points

(ii) closed if starting point = end point.
(also called a loop)

(iii) simple closed curve if it is both simple and closed.

Note:

				
simple	No	Yes	No	Yes
closed	Yes	No	No	Yes

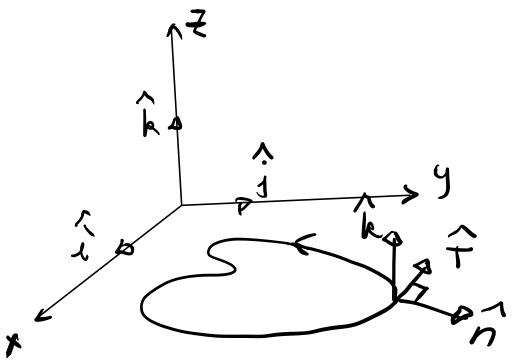
(3) $\vec{F}$ = velocity of fluid

C = oriented plane curve ($C \subset \mathbb{R}^2$)

parametrized by

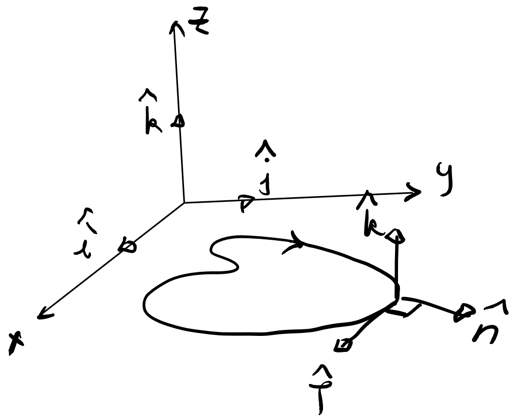
$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$.

$\hat{n}$ = outward-pointing unit normal vector to the curve C.



if C is of anti-clockwise orientation

$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$



if C is of clockwise orientation

$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall $\hat{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \dfrac{x'(t)\hat{i} + y'(t)\hat{j}}{|\vec{r}'(t)|}$

(in arc-length parameter : $\hat{T} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j}$)

Anti-clockwise

$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{x'}{|\vec{r}'|} & \frac{y'}{|\vec{r}'|} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{|\vec{r}'(t)|}$$

(or $\hat{n} = \dfrac{dy}{ds}\hat{i} - \dfrac{dx}{ds}\hat{j}$)

Clockwise

$$\hat{n} = \frac{-y'(t)\hat{i} + x'(t)\hat{j}}{|\vec{r}'(t)|}$$

(or $\hat{n} = -\dfrac{dy}{ds}\hat{i} + \dfrac{dx}{ds}\hat{j}$)

$$\boxed{\text{Flux of } \vec{F} \text{ across } C \overset{\text{def}}{=\!=} \int_C \vec{F}\cdot\hat{n}\, ds}$$

(amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is anti-clockwise parametrization of C.

Then

$$\boxed{\begin{aligned} &\text{Flux of } \vec{F} \text{ across } C \\ &= \oint_C (M\hat{i} + N\hat{j})\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds \\ &= \oint_C M\,dy - N\,dx \end{aligned}}$$

"$\ointctrclockwise$" means the curve is closed and in anti-clockwise orientation.

Similarly "$\ointclockwise$" means the curve is closed and in clockwise orientation.

But in some books, only "$\oint$" is used, no arrow.
Then one needs to know from context whether the parametrization is anti-clockwise or clockwise.

Convention: We usually refer the anti-clockwise orientation as the positive orientation of a closed curve in $\mathbb{R}^2$ (wrt the "orientation" of $\mathbb{R}^2$) and "$\oint$" without orientation explicitly mentioned means positive orientation.

eg 40: Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln: Let $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \leq t \leq 2\pi$

(note: correct orientation!)

(cont'd next time)