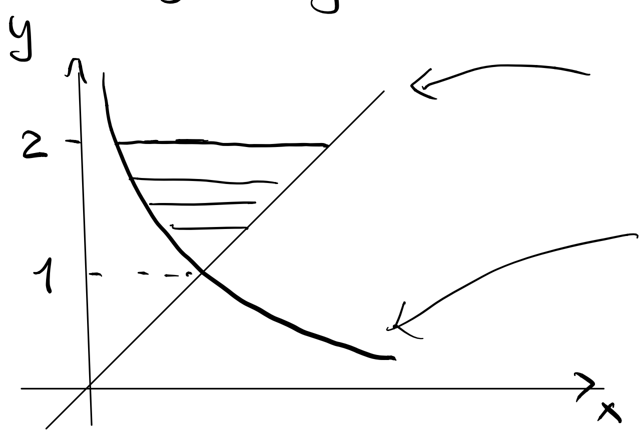
eg30 $I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

Domain of integration



$x = y \leftrightarrow \sqrt{\frac{y}{x}} = 1$

$x = \frac{1}{y} \leftrightarrow \sqrt{xy} = 1$

Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$ (this should simplify the integration)

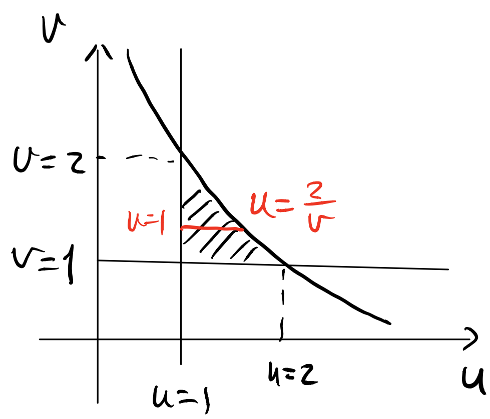
Then

(boundary curves)

$x = y \longleftrightarrow v = 1$

$x = \frac{1}{y} \longleftrightarrow u = 1$

$y = 2 \longleftrightarrow uv = 2$



And the Jacobian (determinant)

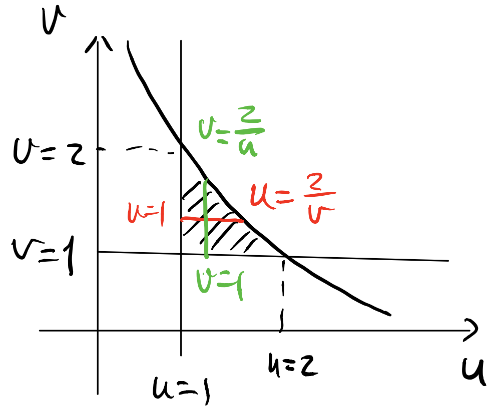
$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Need to express x, y in terms of u, v first:

$uv = y$ & $\frac{u}{v} = x$ i.e. $\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v} \quad \text{(check!)}$$

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$



$$= \int_1^2 \int_1^{\frac{2}{v}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$\text{or } \int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du$$

Let do the $\int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du$

$$= \int_1^2 \int_1^{\frac{2}{u}} \cancel{v} e^u \frac{2u}{\cancel{v}}\, dv\, du$$

$$= \int_1^2 \left[2u e^u \left(\int_1^{\frac{2}{u}} dv\right)\right] du$$

$$= \int_1^2 (4e^u - 2ue^u)\, du \quad \text{(check!)}$$

$$= 2e(e-2) \quad \text{(By integration-by-parts)}$$

#

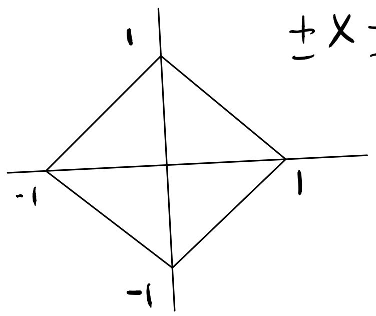
eg 31: Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

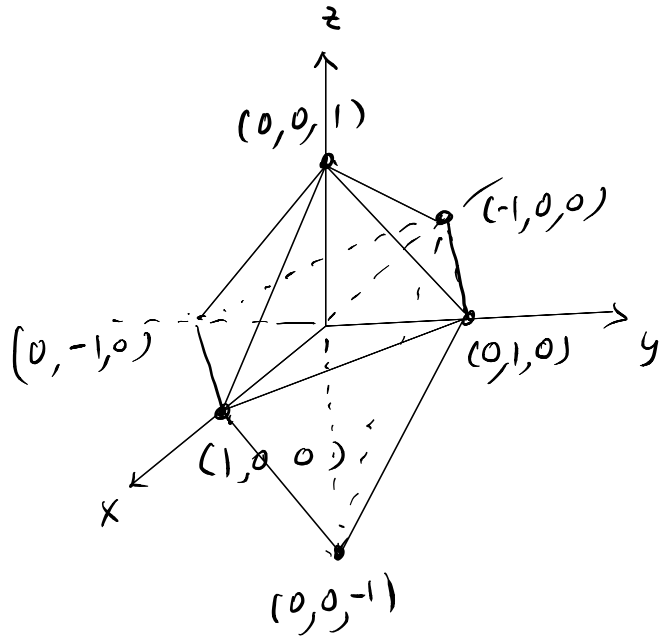
Evaluate $\iiint_D (x+y+z)^4 \, dV$.

Soln: If $z=0$,

then $|x|+|y| \le 1$

$\pm x \pm y = 1$





Boundary surfaces are given

$\pm x \pm y \pm z = 1$

(8 surfaces !)

Let
$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \longleftrightarrow \begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$$

Boundary planes :

$$\pm x \pm y \pm z = 1 \longleftrightarrow \begin{array}{ll} \begin{pmatrix} + & + & + \\ - & - & - \end{pmatrix} & u = \pm 1 \\ \begin{pmatrix} + & + & - \\ - & - & + \end{pmatrix} & v = \pm 1 \\ \begin{pmatrix} + & - & - \\ - & + & + \end{pmatrix} & w = \pm 1 \end{array}$$

$? \quad \begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix} \quad u-v+w=\pm 1$

Change of variable formula $\Rightarrow$

$$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1\le u,v,w\le 1 \\ -1\le u-v+w\le 1}} u^4 \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dv\,dw\,du$$

By $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\right| = \left|-\frac{1}{4}\right| = \frac{1}{4}$

$$\therefore \iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1\le u,v,w\le 1 \\ -1\le u-v+w\le 1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$= A - B - C$$

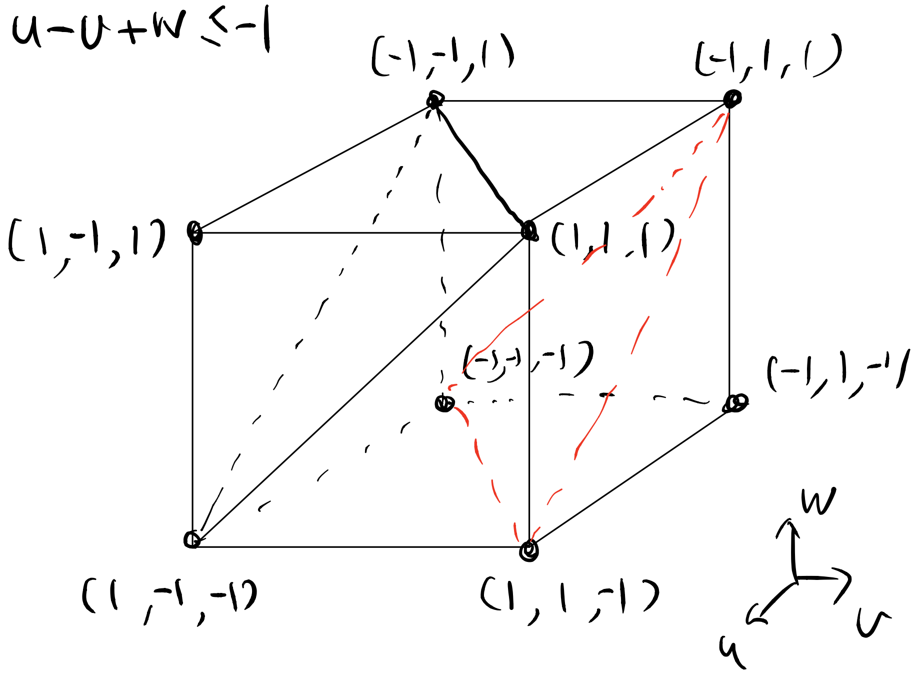
where

$$A = \iiint_{-1\le u,v,w\le 1} \frac{u^4}{4} \, dv\,dw\,du$$

$$B = \iiint_{\substack{-1\le u,v,w\le 1 \\ u-v+w\ge 1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$C = \iiint_{\substack{-1 \le u, v, w \le 1 \\ u - v + w \le -1}} \frac{u^4}{4}\, dv\, dw\, du$$

	$u-v+w$
$(1,-1,1)$	3
$(1,1,1)$	1
$(-1,-1,1)$	1
$(1,-1,-1)$	1



$$\therefore\ A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{u^4}{4}\, dv\, dw\, du \ \underset{\text{easy}}{=\!=} \ \frac{2}{5} \quad (\text{check})$$

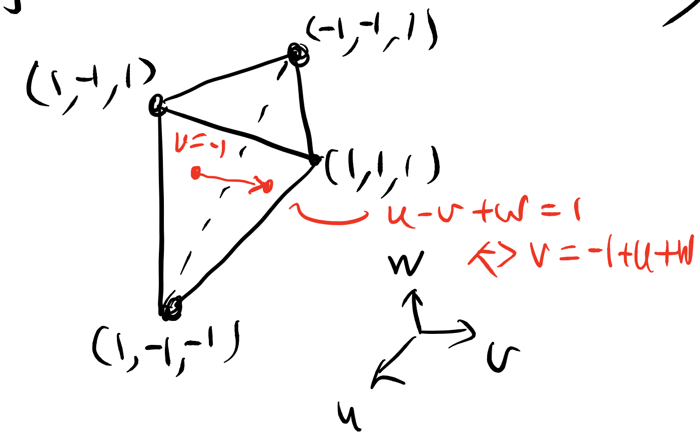
By calculating the values of $u-v+w$ on the vertexes, we see

$$B = \iiint_{\substack{\text{domain determined} \\ \text{by the 4-vertexes} \\ (1,-1,1),\ (1,1,1) \\ (-1,-1,1),\ (1,-1,-1)}} \frac{u^4}{4}\, dv\, dw\, du.$$

By symmetry, the domain for integration C is determined by the other 4 vertexes

$(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$, $(-1,1,1)$

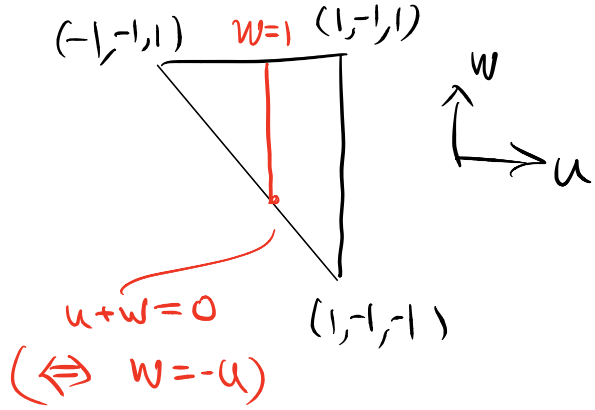
and $C = B$. (by change of variables $(u,v,w) \leftrightarrow (-u,-v,-w)$)



$\therefore$

$$B = \int_{-1}^{1}\int_{-u}^{1}\left(\int_{-1}^{-1+u+w} \frac{u^4}{4}\,dv\right)dw\,du$$

$$= \frac{3}{35} \quad \text{(check!)}$$



Hence $C = \frac{3}{35}$

and

$$\iiint_D (x+y+z)^4\,dV = A - B - C$$

$$= \frac{2}{5} - \frac{3}{35} - \frac{3}{35} = \frac{8}{35}$$

Vecta Analysis

<u>Notation</u>: Usually in textbooks, vectas are denoted by bold-face $\mathbf{i}$, but hard to do it on screen, so my notation fa vectas are:

- general vectas: $\vec{V}$, $\vec{F}$, $\vec{r}$, $\vec{\nabla}$, ... (differential operata)
- unit vectas: $\hat{i}$, $\hat{j}$, $\hat{k}$, $\hat{n}$

Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)
(path integrals)

Def: The <u>line integral</u> of a <u>function</u> f on a <u>curve</u> (path, line) C with parametrization

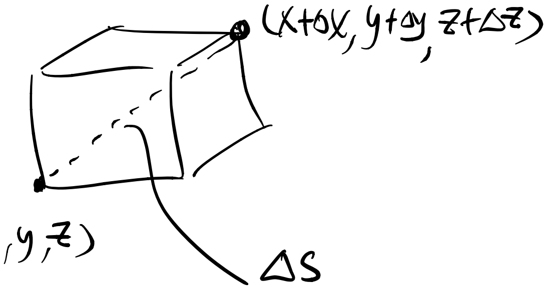
$$\vec{r} : [a,b] \to \mathbb{R}^3$$
$$t \mapsto \vec{r}(t) = (x(t), y(t), z(t))$$

is
$$\int_C f(\vec{r})\,ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i))\,\Delta s_i$$

where P is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

(i.e. ds = length element of a curve)



Remarks:

(1) If $f \equiv 1$, $\boxed{\int_C 1\, ds = \text{arc length of } C}$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

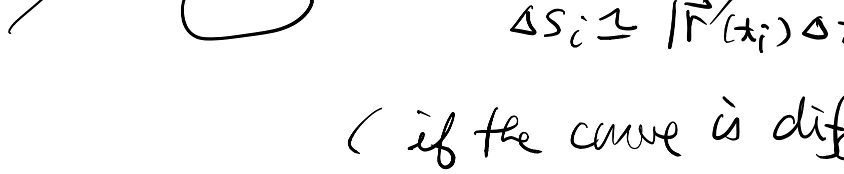
Ref 9' (Formula for line integral)

Notation as in Ref 9, then

$$\int_C f(\vec{r})\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt$$

where $\vec{r}'(t) = (x'(t), y'(t), z'(t))$

Since.



$\Delta s_i \simeq |\vec{r}'(t_i)\Delta t_i| = |\vec{r}'(t_i)|\Delta t_i$

(if the curve is differentiable)

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\,\Delta t_i$$

$$\approx \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\,\Delta t_i$$

$$= |\vec{r}'(t_i)|\,\Delta t_i \quad . \quad \#$$