Case 2 : $\frac{\partial f_1}{\partial x_1}(p)=0$.

Since $\det J(F)=\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}-\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\neq 0$ (at p)

$\Rightarrow \frac{\partial f_1}{\partial x_2}(p)\neq 0 \quad \left(\& \frac{\partial f_2}{\partial x_1}(p)\neq 0\right)$

Interchanging the variables $\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}x_2\\ x_1\end{pmatrix}$.

Then the new mapping $\tilde{F}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}$ satisfies the condition in case 1. Applying case 1 to $\tilde{F}$, then interchanging back to x_1, x_2. #

Step 2 : Let $\begin{pmatrix}y_1\\ y_2\end{pmatrix}=K\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}k(x_1,x_2)\\ x_2\end{pmatrix}$ be a diffeomorphism from region R_1 to $R_2=K(R_1)$. Then for any function $f(y_1,y_2)$ on R_2,

$$\iint_{R_2} f(y_1,y_2)\,dy_1\,dy_2=\iint_{R_1} f\circ K(x_1,x_2)\,|\det J(K)|\,dx_1dx_2$$

$$=\iint_{R_1} f(k(x_1,x_2),x_2)\left|\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right|dx_1dx_2$$

Pf of Step 2: By additivity property of integrations and cutting R_1 (and correspondingly $R_2=K(R_1)$) into small regions, we may assume $R_1=[a,b]\times[c,d]=\{a\le x_1\le b,\ c\le x_2\le d\}$

For any fixed $y_2=x_2$, $y_1=k(x_1,x_2)=k(x_1,y_2)$, for $a\le x_1\le b$, can be regarded as a transformation of 1-variable.

Note that $\frac{\partial y_1}{\partial x_1} = \frac{\partial k}{\partial x_1} = \det\begin{pmatrix} \frac{\partial k}{\partial x_1} & \frac{\partial k}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \det J(K) \neq 0$

(Since K is a diffeomorphism)

Note also that R_2 is a special type:

$$\{ c \leq y_2 \leq d,\ k(a,y_2) \leq y_1 \leq k(b,y_2) \} \quad \left(\frac{\partial y_1}{\partial x_1} > 0\right)$$

or $$\{ c \leq y_2 \leq d,\ k(b,y_2) \leq y_1 \leq k(a,y_2) \} \quad \left(\frac{\partial y_1}{\partial x_1} < 0\right)$$

By Fubini's Thm (assuming $\frac{\partial y_1}{\partial x_1} > 0$, the other case is similar)

$$\iint_{R_2} f(y_1,y_2)\, dy_1 dy_2 = \int_c^d \left(\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\, dy_1 \right) dy_2$$

and Change of variable formula in 1-variable implies

$$\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\, dy_1 = \int_a^b f(k(x_1,x_2), y_2) \frac{\partial y_1}{\partial x_1}\, dx_1$$

$$= \int_a^b f(k(x_1,x_2), x_2) \frac{\partial y_1}{\partial x_1}\, dx_1$$

$$\Rightarrow \iint_{R_2} f(y_1,y_2)\, dy_1 dy_2 = \int_c^d \left(\int_a^b f(k(x_1,x_2), x_2) \left| \frac{\partial y_1}{\partial x_1} \right| dx_1 \right) dx_2$$

(since $\frac{\partial y_1}{\partial x_1} > 0$)

$$= \int_c^d \int_a^b f(k(x_1,x_2), x_2)\, |\det J(K)|\, dx_1 dx_2$$

$$= \iint_{R_1} f(k(x_1,x_2), x_2)\, |\det J(K)|\, dx_1 dx_2 \quad \#$$

Note: The Step 2 is also true for $K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1, x_2) \\ x_1 \end{pmatrix}$

and $H\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix}$.

Step 3: If the change of variables formula holds for F & G, then it holds for $F \circ G$

Pf: Easily by $J(F \circ G) = J(F)\, J(G)$ (Chain rule)

$\Rightarrow \quad |\det J(F \circ G)| = |\det J(F)|\,|\det J(G)|$. #

Final Step: Combining steps 1–3, and using additivity property of integration, we've proved the Thm 6 for general change of variable formula. #

[Actually, this applies to all dimensions.]

Substitutions in triple integrals

$$\phi(u, v, w) = (x, y, z) \quad : \quad G \to D, \quad G \subset \mathbb{R}^3, \; D \subset \mathbb{R}^3$$

with $\begin{cases} x = g(u, v, w) \\ y = h(u, v, w) \\ z = k(u, v, w) \end{cases}$ 1-1, onto, differentiable & inverse also differentiable.

Ref 8 Jacobian (determinant) of transformation in $\mathbb{R}^3$.

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{pmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Note: Chain rule $\Rightarrow$

2-dim $\quad \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(s,t)}$

3-dim $\quad \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(s,t,r)} = \frac{\partial(x,y,z)}{\partial(s,t,r)}$

$\Rightarrow$

2-dim $\quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$

3-dim $\quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$

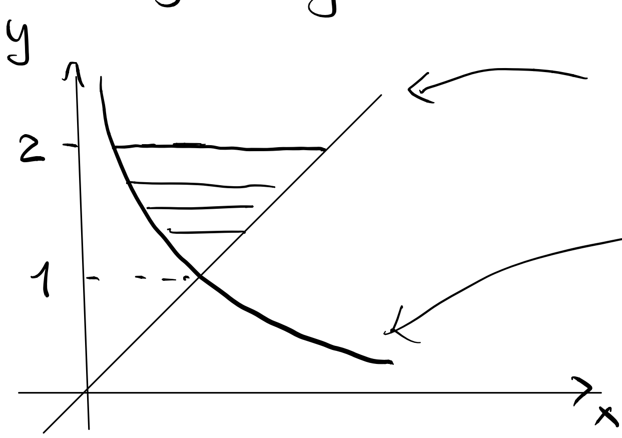
Thm 7: Under similar conditions of Thm 6

$$\iiint_D F(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_G F(\phi(u,v,w))\,|J(u,v,w)|\,du\,dv\,dw$$

$$= \iiint_G F(g(u,v,w),\, h(u,v,w),\, k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

eg30 $I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

Domain of integration



$x = y \leftrightarrow \sqrt{\frac{y}{x}} = 1$

$x = \frac{1}{y} \leftrightarrow \sqrt{xy} = 1$

Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$ (this should simplify the integration)