Change of Variables Formula

(Substitution in multiple integrals)

Review of 1-variable

$$\int_a^b f(x)\,dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du$$

$x = x(u)$ for $u \in [c,d]$

provided $\frac{dx}{du} > 0$ ($\Rightarrow c < d$)

and $\int_a^b f(x)\,dx = \int_d^c f(x(u))\frac{dx}{du}\,du$, if $\frac{dx}{du} < 0$

($\Rightarrow c > d$)

Recall, in Riemann sum (of general dimensions):

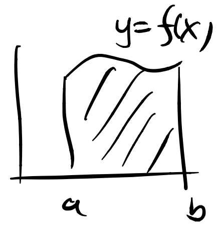
$$\int_{[a,b]} f(x)\,dx$$

$dx \sim |\Delta x|$ (length of Δx !)

$[a,b]$ as set (we don't care about the direction!)

$\therefore$ we actually have

$$\int_a^b f(x)\,dx = \begin{cases} \int_{[a,b]} f(x)\,dx & \text{if } a \le b \\ -\int_{[a,b]} f(x)\,dx, & \text{if } a \ge b \end{cases}$$



Combining these,

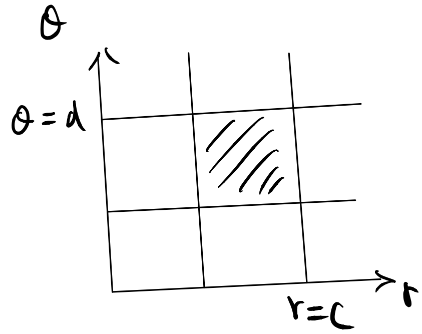
$$\boxed{\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x)\left|\frac{dx}{du}\right| du}$$

(since $\frac{|\Delta x|}{|\Delta u|} \sim \left|\frac{dx}{du}\right|$.)

$$\left(\frac{dx}{du} < 0 \Rightarrow c > d \Rightarrow \int_{[c,d]} f(x)\left|\frac{dx}{du}\right| du = -\int_d^c f(x)\left|\frac{dx}{du}\right| du = \int_d^c f(x)\frac{dx}{du}\, du \right)$$

Back to multiple integrals

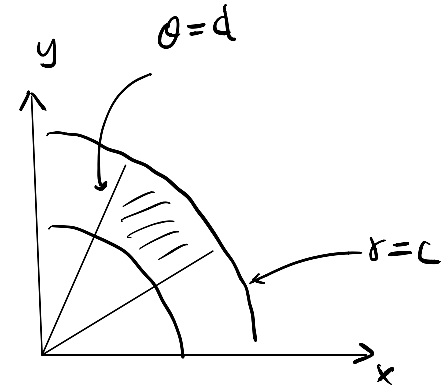
Recall: Polar coordinates

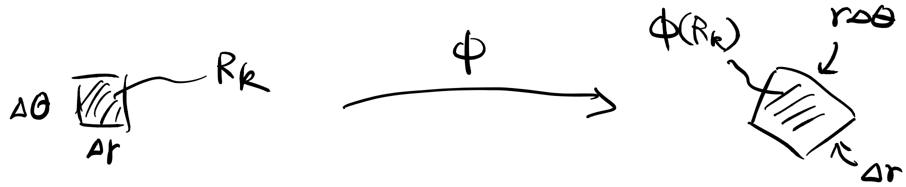


$$\xrightarrow{\ \phi\ }$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\boxed{\phi(r,\theta) = (x,y)}$$





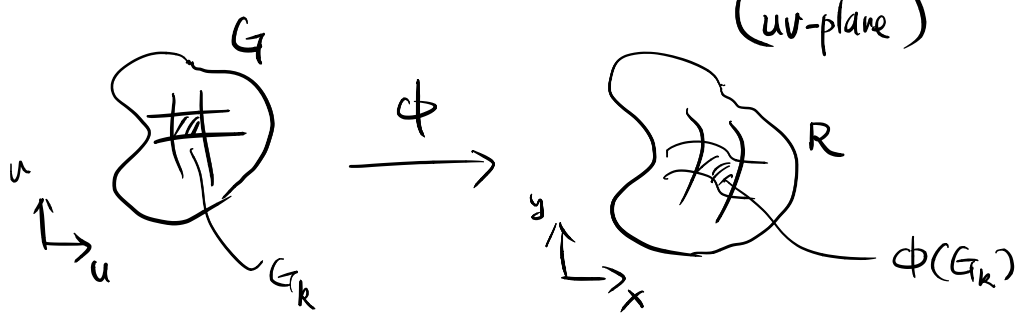
$$\text{Area}(R_k) \cong \Delta r \Delta\theta \qquad \text{Area}(\phi(R_k)) \cong r\Delta r\Delta\theta$$

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

General change of coordinates formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ denoted by $\phi(u,v) = (x,y)$

$$\underset{(\text{uv-plane})}{\phi : G} \longrightarrow R \ (\subset \text{xy-plane})$$



Idea: We need to find

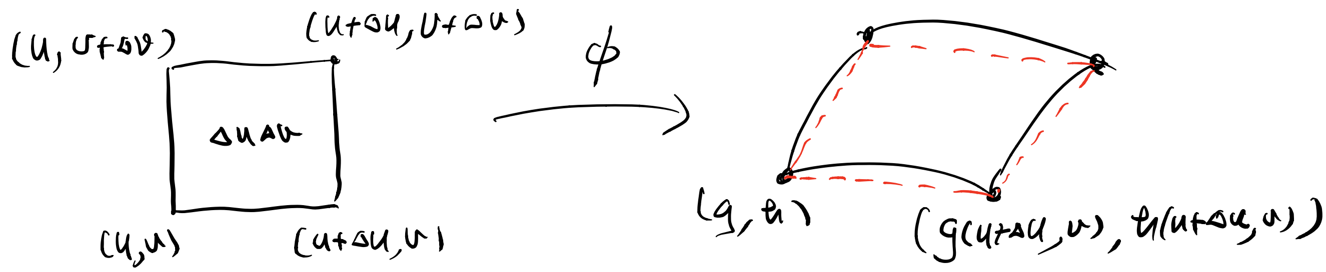
$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \longrightarrow \, ? \quad \text{as "} G_k \to \text{point"}$$

If ϕ is $\left(\text{diffeomorphism: 1-1, onto \& } \phi, \phi^{-1} \in C^1\right)$

C^1, then

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$



The parallelogram is "approximately" given by the linear transformation

$$\begin{pmatrix} \Delta g \\ \Delta h \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \\ \dfrac{\partial h}{\partial u} & \dfrac{\partial h}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

(By linear algebra)

$$\Longrightarrow \quad \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \sim \frac{\Delta g \Delta h}{\Delta u \Delta v} \sim \left| \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \right|$$

Def 7 Define the Jacobian $J(u,v)$ of the "coordinate" transformation

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

by

$$J(u,v) \underset{\text{notation}}{=} \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

With this notation, we should have the formula:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_G f(g(u,v), h(u,v)) \, |J(u,v)| \, du\,dv$$

eg 28:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\Rightarrow \quad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = r \quad \text{(check!)}$$

and

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta$$

$$= \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\,d\theta$$

(same formula as before)

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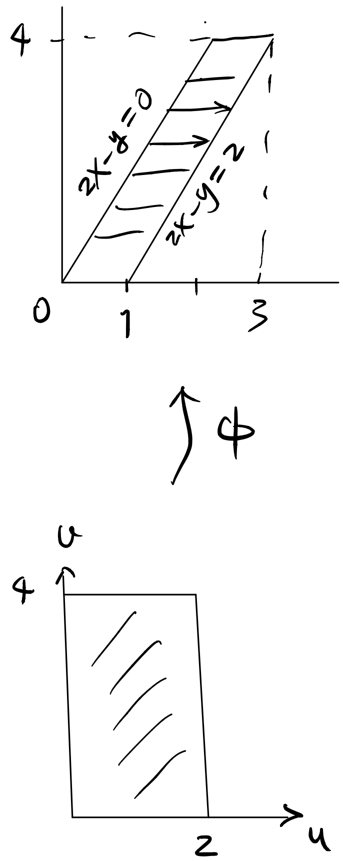
eg 30 $\displaystyle\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$

lower limit $x = \frac{y}{2} \Leftrightarrow 2x-y=0$

upper limit $x = \frac{y}{2}+1 \Leftrightarrow 2x-y=2$



Define $\begin{cases} u = 2x-y \\ v = y \end{cases}$

Then $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x-y=0 \longleftrightarrow u=0 \\ 2x-y=2 \longleftrightarrow u=2 \end{cases}$

$\begin{cases} y=0 \longleftrightarrow v=0 \\ y=4 \longleftrightarrow v=4 \end{cases}$

$$J(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_0^4 \int_0^2 \frac{u}{2}\cdot\left|\frac{1}{2}\right|\,du\,dv$$

$= 2$ (Check!)

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Thm 6 Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, ϕ & $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is continuous on R, then

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Notes: (i) $f\circ\phi(u,v) = f(x(u,v), y(u,v))$

(ii) ϕ is a diffeomorphism $\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$.

Pf of Thm 6

Step 0: We need better notations and terminology:

In this proof, we'll denote

$$J(\phi) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \quad \text{the Jacobian matrix.}$$

and $$\frac{\partial(x,y)}{\partial(u,v)} = \det J(\phi) \quad \text{the Jacobian determinant.}$$

- We also use "index" notations for variables:

$$(x_1, x_2) \text{ or } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \left(\text{instead of } (x,y), \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Step 1 : Let $F : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$ near a point p

with $\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} \neq 0$ at p. Then, near the point p, F can be

decomposed into $F = H \circ K$

with H, K of the forms

$$K : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} k(x_1, x_2) \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\left(\text{or} \quad \begin{pmatrix} k(x_1, x_2) \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$$

and

$$H : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix}$$

such that $\det J(K) \neq 0$ and

$\det J(H) \neq 0$.

Pf of Step 1 : By assumption $0 \neq \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$ (at p)

Case 1 : $\frac{\partial f_1}{\partial x_1}(p) \neq 0$

Define $k(x_1, x_2) = f_1(x_1, x_2)$ near p.

Then the transformation

$$K : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 = k(x_1, x_2) = f_1(x_1, x_2) \\ y_2 = x_2 \end{pmatrix}$$

is of the required form and has Jacobian matrix

$$J(K) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det J(K)(p) = \frac{\partial f_1}{\partial x_1}(p) \neq 0$$

By Inverse Function Theorem, K is invertible near p and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g(y_1, y_2) \\ y_2 \end{pmatrix}$$ is differentiable at $K(p)$ (since $x_2 = y_2$)

with $\quad J(K^{-1})_{K(p)} \cdot J(K)_p = Id.$

i.e.
$$\begin{pmatrix} \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_1} = 1 \quad \& \quad \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_2} + \frac{\partial g}{\partial y_2} = 0 .$$

In particular, $\det J(K^{-1})_{K(p)} = \dfrac{1}{\det J(K)_p} \neq 0$.

Now, define

$$h(y_1, y_2) = f_2 \circ K^{-1}(y_1, y_2)$$
$$= f_2(g(y_1, y_2), y_2) \qquad (= f_2(x_1, x_2))$$

and
$$H : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 = y_1 \\ z_2 = h(y_1, y_2) \end{pmatrix}$$

(is of the required form)

Moreover $\quad J(H) = \begin{pmatrix} 1 & 0 \\ \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \end{pmatrix}$

$$\Rightarrow \quad \det J(H) = \frac{\partial h}{\partial y_2} = \frac{\partial f_2}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial y_2} \quad \text{(Chain rule)}$$
$$= \frac{\partial f_2}{\partial x_1}\frac{\partial g}{\partial y_2} + \frac{\partial f_2}{\partial x_2} \cdot 1$$

$$= \frac{\partial f_2}{\partial x_1}\left(-\frac{\partial f_1}{\partial x_2}\frac{\partial g}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}$$

$$= -\frac{\frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}}{\frac{\partial f_1}{\partial x_1}} + \frac{\partial f_2}{\partial x_2}$$

$$= \frac{1}{\frac{\partial f_1}{\partial x_1}}\left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right] \qquad \frac{\partial(f_1,f_2)}{\partial(x_1,x_2)}$$

$$= \frac{1}{\frac{\partial f_1}{\partial x_1}} \det J(F) \neq 0 \quad \text{at } p.$$

So, H & K satisfy the requirements and we have

$$H \circ K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This completes case 1.