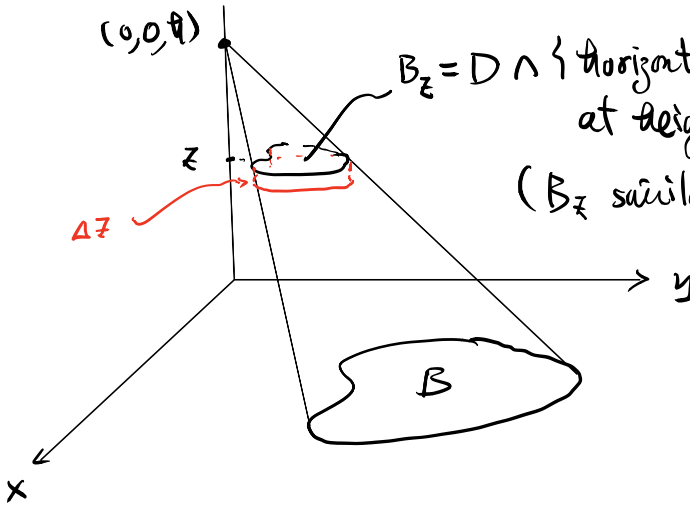
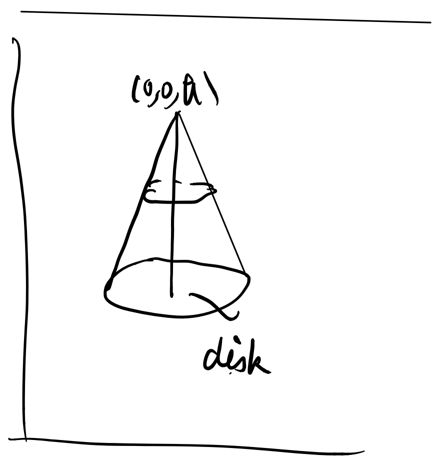
eg 22 : Let B (base) be a "nice" subset of $\mathbb{R}^2$

Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$



$B_z = D \cap \{$horizontal plane at height $z\}$

(B_z similar to B)



How to find volume of D ?

$\text{Vol} \sim \text{Area}(B_z)\cdot \Delta z$

Answer : by concept of Riemann sum and this figure

$$\text{Vol}(D) = \int_0^h \text{Area}(B_z)\,dz$$

By similarity : ratio of height : $\frac{h-z}{h} = 1-\frac{z}{h}$

$\Rightarrow$ ratio of the area : $\frac{\text{Area}(B_z)}{\text{Area}(B)} = \left(1-\frac{z}{h}\right)^2$

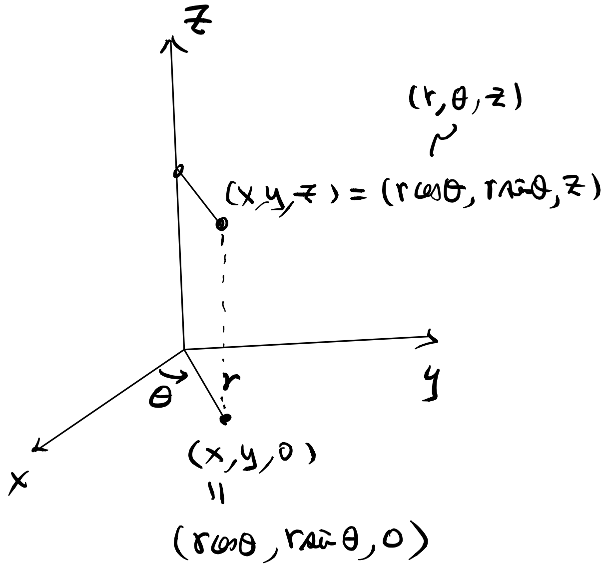
$$\Rightarrow \text{Vol}(D) = \int_0^h \left(1-\frac{z}{h}\right)^2 \text{Area}(B)\,dz$$

$$= \int_0^h \left(1-\frac{z}{h}\right)^2 dz \cdot \text{Area}(B)$$

$$= \frac{h}{3}\,\text{Area}(B) \quad \#$$

(check!)

Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane
  $(r \geq 0)$
- z = rectangular vertical coordinates.

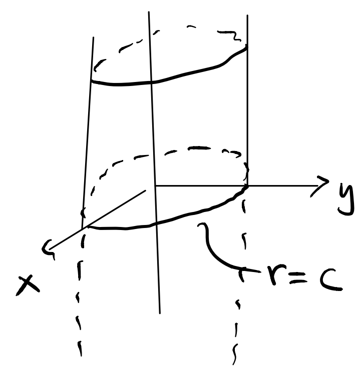
Then a point $P: (x,y,z)$ can be represented by (r,θ,z) where

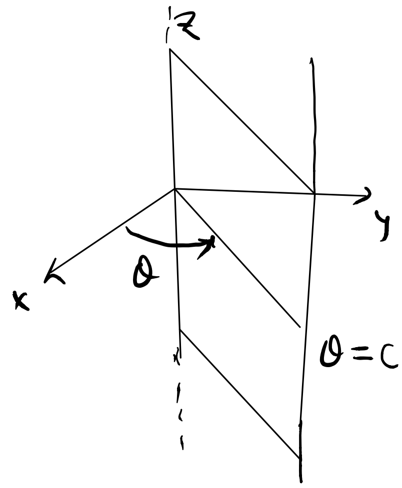
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

And (r,θ,z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$

<u>Remark 1</u>: (c is a constant)

- $r = c$ $(c > 0)$ describes a <u>cylinder</u>
- $\theta = c$ describes a
- $z = c$ describes a horizontal plane
  (as in rectangular coordinates)





<u>Remark 2</u>: We can define cylindrical coordinates in other directions:

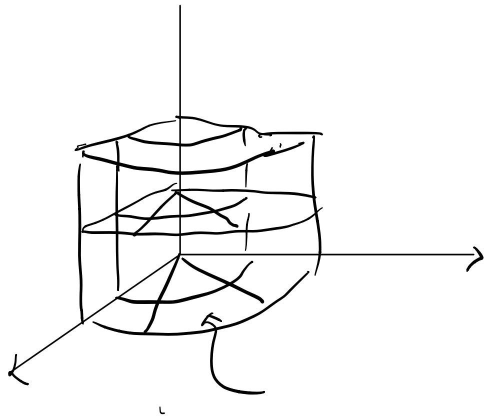
eg:

$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$$

Volume element

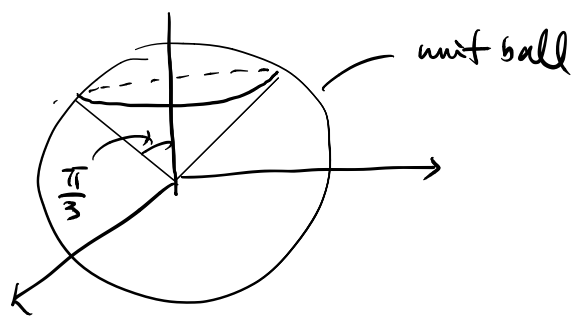
$$dV = \underbrace{dx\,dy}\,dz$$
$$= r\,dr\,d\theta \;\; dz$$

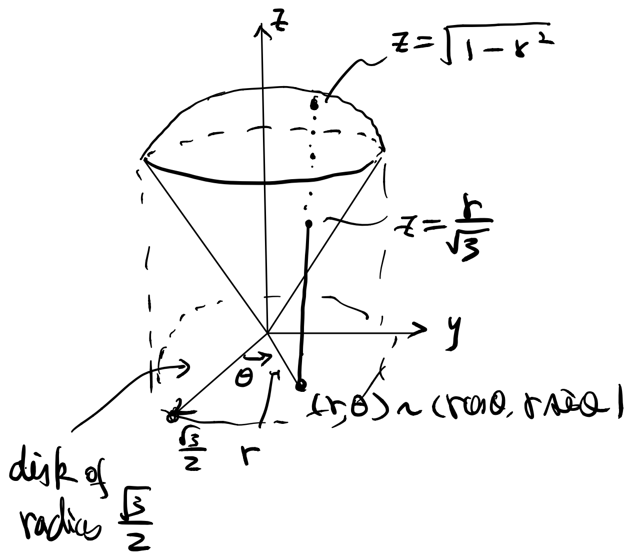
(order of the integration can be changed)



eg 23 (see also eg 25)

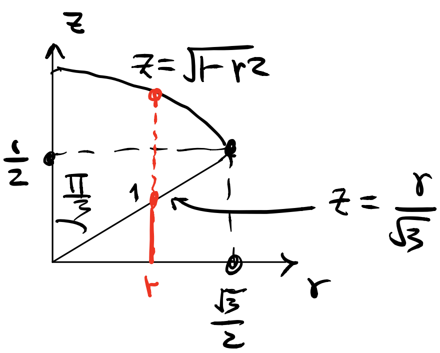
Find the volume of the Ice-cream cone I given in the figure





Soln:

θ fixed



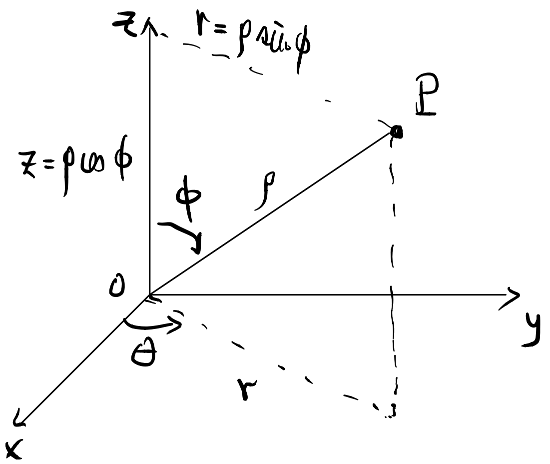
Fubini's

$$\Rightarrow \quad Vol(D) = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} r\left(\sqrt{1-r^2}-\frac{r}{\sqrt{3}}\right)dr\,d\theta = \frac{\pi}{3} \quad \text{(check!)}$$

#

Spherical coordinates in $\mathbb{R}^3$



$(\rho = \text{rho})$

(ρ, ϕ, θ) where

- ρ = distance from the origin $(\rho \geq 0)$
- ϕ = angle from the positive z-axis to $\overline{OP}$ $(0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$

Remark: If (r, θ, z) is the cylindrical coordinates of P,

then $\begin{cases} r = \rho \sin\phi \\ z = \rho\cos\phi \end{cases}$

In particular $z^2 + r^2 = \rho^2$

Then

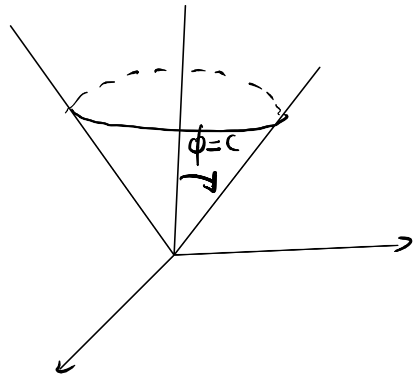
$$\boxed{\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = z = \rho\cos\phi \end{cases}}$$

rectangular cylindrical spherical

Remark: If c is a constant, then

- $\rho = c$ $(c > 0)$ describes a sphere of radius c
- $\theta = c$ describes a half-plane (vertical)
- $\phi = c$ describes

$$\begin{cases} \text{+ve } z\text{-axis} & \text{if } c = 0 \\ \text{-ve } z\text{-axis} & \text{if } c = \pi \\ xy\text{-plane} & \text{if } c = \frac{\pi}{2} \\ \text{cone} & \text{otherwise} \end{cases}$$



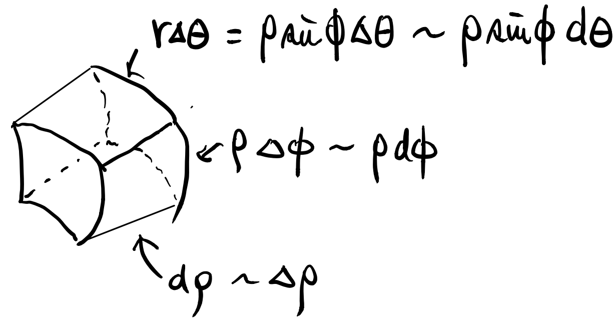
Volume element

$$dV = \underbrace{dx\,dy}\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho\sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$

(ρ, ϕ) is the polar of (z, r) !

$$= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



eg 24: Convert the following into spherical coordinates

(1) $x^2 + y^2 + (z-1)^2 = 1$ (sphere)

(2) $z = -\sqrt{x^2+y^2}$ (cone)

Soln: (1) sub. $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

into $x^2 + y^2 + (z-1)^2 = 1$

$$\Rightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Rightarrow \quad \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\Rightarrow \quad \rho^2 - 2\rho\cos\phi = 0$$

i.e. $\rho(\rho - 2\cos\phi) = 0$ together with $\rho \geq 0$

$\Rightarrow \quad \rho = 2\cos\phi$ ($\rho = 0$ is a point only)

(2) sub. the formula into $z = -\sqrt{x^2+y^2}$

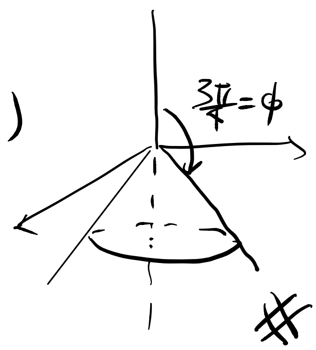
$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi$ (since $x^2+y^2 = r^2 = \rho^2\sin^2\phi$)

($\rho \geq 0$; $0 \leq \phi \leq \pi \Rightarrow \sin\phi \geq 0$)

For $\rho \neq 0$ (i.e. not the origin)

$\cos\phi = -\sin\phi$ ($\tan\phi = -1$)

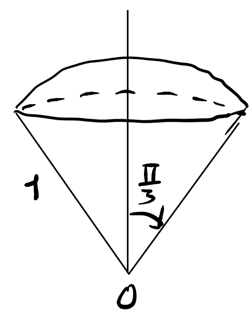
$\Rightarrow \quad \phi = \frac{3\pi}{4}$



#

<u>eg25</u> (see eg23)

Volume of ice-cream cone again in spherical coordinates



<u>Soln</u>: The ice-cream cone I is given by

$$\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{3} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

don't miss this!

$$\Rightarrow \quad Vol(I) = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

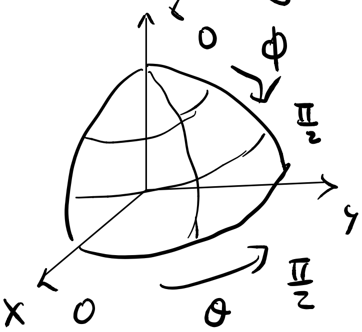
(much easier than cylindrical) $= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}}\sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right) = \frac{\pi}{3}$ (check!)

#

eg 26: $f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

(f is continuous ; and in fact, we don't need this as value of f at one point doesn't affect the $\iiint_D f\,dV$.)

Let D = unit ball centered at origin intersecting with the 1^{st} octant.



Then D can be represented in spherical coordinates:

$$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

And $(x,y,z) \neq 0$ $\quad f(x,y,z) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \dfrac{(\rho\sin\phi)^2}{\rho} = \rho \sin^2\phi$

(as $\rho \to 0$, $f \to 0$ $\therefore$ f is continuous, and the formula is correct even for $(x,y,z) = 0$.)

Hence $\displaystyle\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\overset{f}{\rho\sin^2\phi})\ \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$

$$= \frac{\pi}{2}\left(\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi\right)\left(\int_0^1 \rho^3 d\rho\right)$$

$$= \frac{\pi}{12} \quad \text{(check!)}$$

If we want to calculate the <u>average</u> of f over D, we need to calculate Vol(D) too.

In our case $\mathrm{Vol}(D) = \frac{1}{8}\,\mathrm{Vol}(\text{unit sphere}) = \frac{1}{8}\cdot\frac{4\pi}{3} = \frac{\pi}{6}$

Hence

$$\text{average of } f \text{ over } D = \frac{1}{\mathrm{Vol}(D)}\cdot\iiint_D f(x,y,z)\,dV$$

$$= \frac{1}{2}\,.\ \cancel{\times}$$

<u>eg</u> 27 (Improper integrals)

Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$ (unbounded as $\rho \to 0$)

$g(x,y,z) = \dfrac{1}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \dfrac{1}{\rho^3}$

over unit ball $B = \{(\rho,\phi,\theta) : 0 \le \rho \le 1\}$

(i) Does $\displaystyle\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV$ exist ?

where $B_\varepsilon = \{(\rho,\phi,\theta) : 0 \le \rho \le \varepsilon\}$

(ii) Does $\displaystyle\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV$ exist?