Applications

(1) Area (of (good) region $R \subset \mathbb{R}^2$)

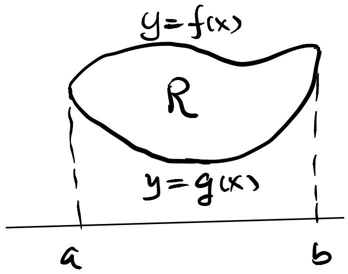
Def 3 $\quad \text{Area}(R) = \iint_R 1 \, dA$

Then Fubini's Thm implies the well-formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)] \, dx$$

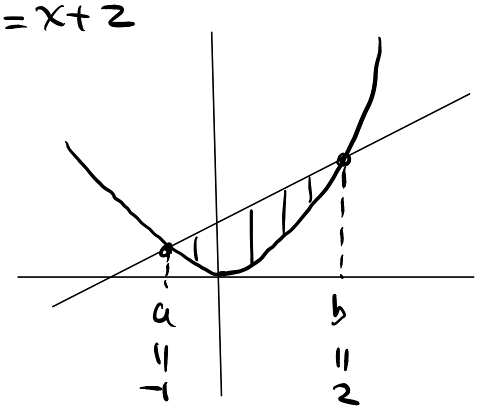
if R is the region bounded by the curves $y = f(x) \geq g(x) = y$.

($f(a) = g(a)$ & $f(b) = g(b)$) for $a \leq x \leq b$



(Ex!)

eg 10: Area bounded by $y = x^2$ and $y = x + 2$



Soln: Solving $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

$\Rightarrow \quad x^2 = x + 2$

$\Rightarrow \quad x = -1, 2$

Then by Fubini's

$$\text{Area} = \int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy \, dx = \int_{-1}^{2} [(x+2) - x^2] \, dx = \frac{9}{2}$$

(Check!)

(2) Average (of a function over a region)

Let $f : R \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an integrable function.

Def 4: The average value of f over R

$$= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

eg 11: Let $f(x,y) = x\cos(xy)$, $R = [0,\pi]\times[0,1]$

Find average of f over R.

Soln: Average of f over $R = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

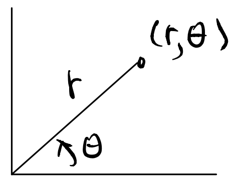
$$= \frac{1}{\pi}\int_0^{\pi}\int_0^1 x\cos(xy)\,dy\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx \qquad \text{(check!)}$$

$$= \frac{2}{\pi} \qquad \text{(check!)} \qquad \#$$

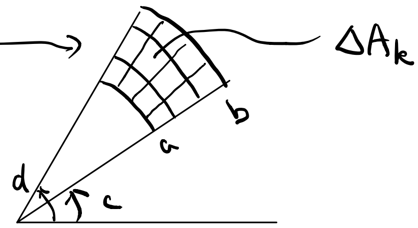
Double integral in polar coordinates

$(r, \theta) \longleftrightarrow (x, y)$



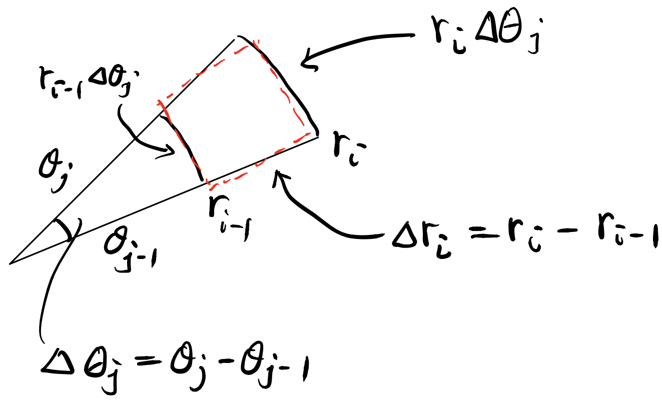
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases}$$



<u>Idea</u>: $\sum_k f(\text{point}_k)\,\Delta A_k$

what is ΔA_k (approximately) ?



$$\therefore \quad \Delta A_k \simeq (r_i \Delta\theta_j)\,\Delta r_i \simeq (r_{i-1}\Delta\theta_j)\,\Delta r_i$$

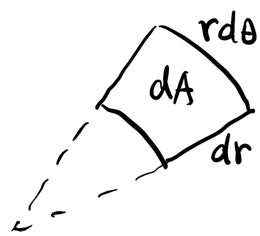
Hence $\quad \Delta A_k \simeq \Delta x \Delta y \simeq \underline{r\,\Delta\theta}\,\Delta r$

So $\quad \displaystyle\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$

$$= \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Method to remember the formula

$$\underline{dA = dx\,dy = r\,dr\,d\theta}$$



Double integral of f over $R = \{(r,\theta) : a \leq r \leq b,\ c \leq \theta \leq d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_c^d \left[\int_a^b f(r,\theta)\, r\,dr\right] d\theta$$

$$= \int_a^b \left[\int_c^d f(r,\theta)\,d\theta\right] r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$.

Remark: This is a special case of the change of variables formula. The "extra" factor "r" in the integrand is in fact

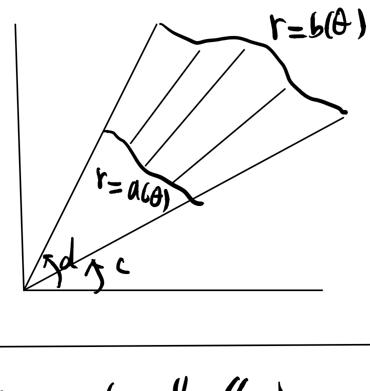
$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$ the Jacobian determinant of the change of coordinates.

More generally

Thm 3 If R is a (closed and bounded) region with $c \leq \theta \leq d$ and $a(\theta) \leq r \leq b(\theta)$ $\big(0 \leq a(\theta) \leq b(\theta),\ a(\theta) \not\equiv b(\theta)\big)$

And $f : R \to \mathbb{R}$ is a continuous function on R, then

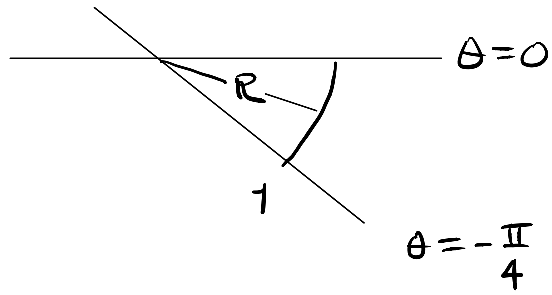
$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\right] d\theta$$



(remember this extra "r")

eg 12: Back to our previous example 9,

$f(x,y) = x = r\cos\theta$



and hence

$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\int_0^1 r\cos\theta \; r\,dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\cos\theta \int_0^1 r^2 dr \right] d\theta = \int_{-\frac{\pi}{4}}^{0} \frac{1}{3}\cos\theta \, d\theta = \frac{1}{3\sqrt{2}} \quad \text{(Check!)}$$

(Much easier than before !)