Furthermore, we have

Prop 3: Let $R=[a,b]\times[c,d]$ be a closed rectangle, $f(x,y)$ and $g(x,y)$ be functions on R, and $k\in\mathbb{R}$ is a constant.

(1) If f and g are integrable over R, then $f\pm g$ and kf are integrable over R.

(2) In the case of (1), we have

$$\iint_R [f\pm g](x,y)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

and

$$\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA.$$

Pf: Omitted (Obvious from the concept of Riemann sum)

Remark: This Prop 3 implies that the set of integrable functions over R forms a "vector space over $\mathbb{R}$", and "(double) integral" is linear (when the rectangle R is fixed).

Prop 4: (a) If $f(x,y)\geq 0$ is an integrable function on a closed rectangle R, then

$$\iint_R f(x,y)\,dA \geq 0.$$

(b) If R_1 and R_2 be two closed rectangles such that $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$, then

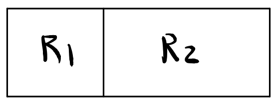
$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

for integrable function f over $R_1 \cup R_2$

Pf: Omitted (Obvious from the concept of Riemann sum)

Note: Various situation for $\text{int } R_1 \cap \text{int } R_2 = \emptyset$

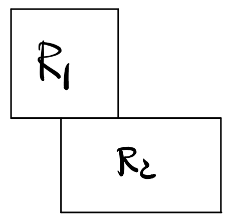
(1)



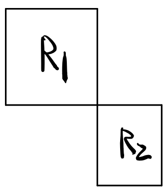
$R_1 \cap R_2 =$ common edge

$\text{int} R_1 \cap \text{int} R_2 = \emptyset$

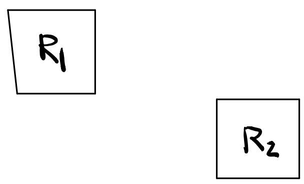
(2)



(3)



(4)

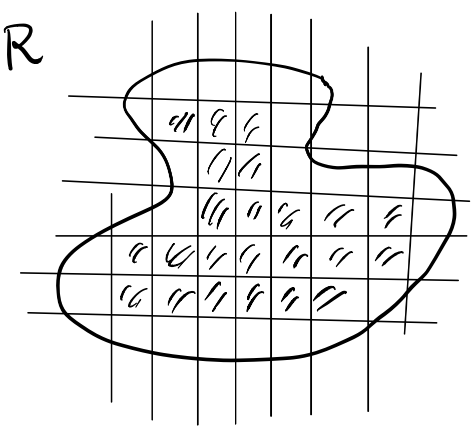


We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) − (4).

Hence we need to define double integrals over general regions.

Double Integrals over General Regions

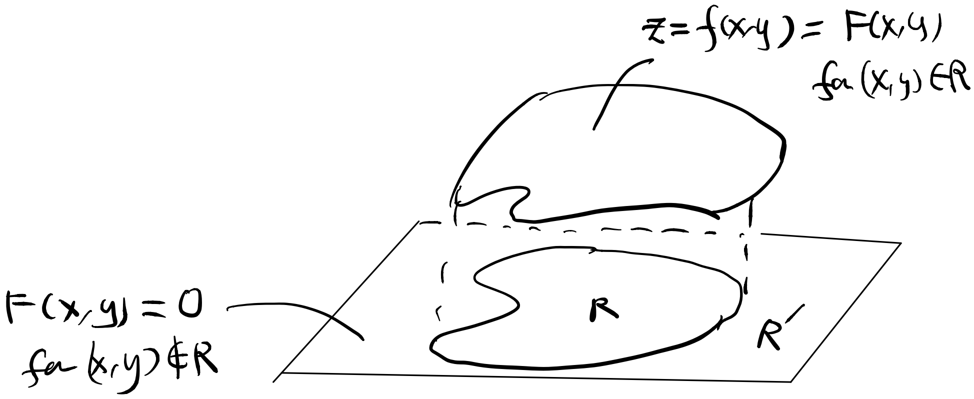
For non-rectanglar bounded (closed) region R one can define similarly the concept of "Riemann sum". There are two ways to form the "Riemann sum":



(i) sum over all subrectangles completely inside R

(ii) sum over all "subrectangles" with non-empty intersection with R.

Or define as follows:



R' = "large" rectangle containing R

<u>Def 2</u>: Let R be a <u>bounded region</u> and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in R \\ 0, & \text{if } (x,y) \in R' \setminus R. \end{cases}$$

Then the <u>integral of f over R</u> is defined by

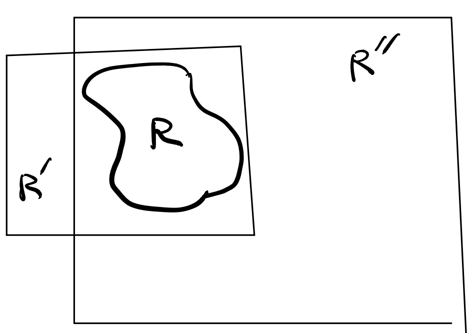
$$\iint_R f(x,y)\, dA = \iint_{R'} F(x,y)\, dA$$

<u>Remark</u>: The definition is well-defined (i.e. doesn't depend on the choice of R'): if R'' is another rectangle s.t. $R'' \supset R$

and $$\tilde{F}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in R \\ 0 & \text{if } (x,y) \in R'' \setminus R. \end{cases}$$

Then (by Prop 4(b))

$$\iint_{R''} \tilde{F}(x,y)\, dA = \iint_{R'} F(x,y)\, dA.$$



Prop 5: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region".

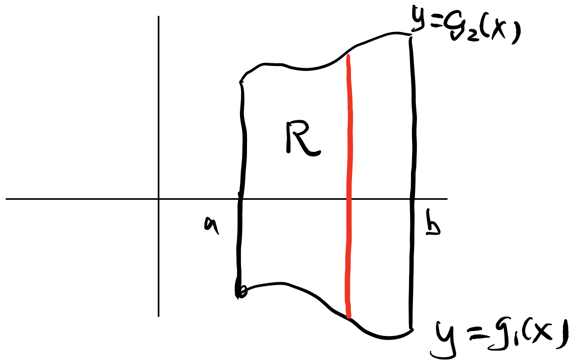
(together with the remark (i) of Prop 2)

Important special types of bounded regions R

Type (1) $R = \{(x,y): a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

where g_1 and g_2 are "continuous" functions on $[a,b]$

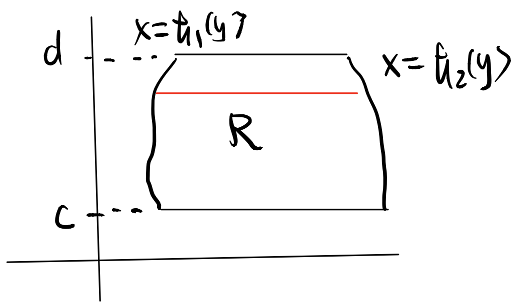
($g_1 \le g_2$ but not $g_1 \not\equiv g_2$)



Type (2) $R = \{(x,y): h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$

($h_1 \le h_2$, but not $h_1 \not\equiv h_2$)



For these 2 types of bounded regions, we have

Thm 2 (Fubini's Theorem (Stronger version))

Let $f(x,y)$ be a continuous function on a closed and bounded region R.

(1) If R is of type (1) as above, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

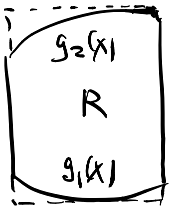
(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Pf: Type (1): Extend $f(x,y)$ to $F(x,y)$ as above on a rectangle

$R' = [a,b] \times [c,d]$ such that

$c = \min_{[a,b]} g_1(x)$ and $d = \max_{[a,b]} g_2(x)$.



By definition 2,

$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

$$\overset{\text{Fubini 1st form}}{=} \int_a^b \left[\int_c^d F(x,y)\,dy\right] dx$$

[f continuous on $R \Rightarrow F$ continuous on R' except possibly on the boundary curves of R. Hence by remark (ii) of Prop 2, F is integrable over R'. And the Fubini theorem (1st form) is in fact true for integrable functions on a rectangle.]

Now note that $F(x,y) = 0$ for $y < g_1(x)$ or $y > g_2(x)$

and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

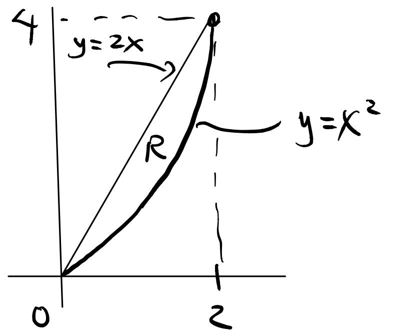
$$\therefore \iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \qquad \#$$

Type (2) can be proved similarly.

eg7 Integrate $f(x,y) = 4y+2$

over the region bounded by $y = x^2$ and $y = 2x$

Soln:



By Fubini's

$$\iint_R f(x,y)\,dA$$

$$= \int_0^2 \left[\int_{x^2}^{2x} (4y+2)\,dy \right] dx$$

$$= \int_0^2 (-2x^4 + 6x^2 + 4x)\,dx \quad \text{(check!)}$$

$$= \frac{56}{5} \quad \text{(check!)}$$

In fact, R is also type (2), and Fubini's

$$\Rightarrow \quad \iint_R f(x,y)\,dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\,dx\,dy$$

$$= \int_0^4 (4y+2)\left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$= \cdots = \frac{56}{5} \quad \text{(check!)}$$

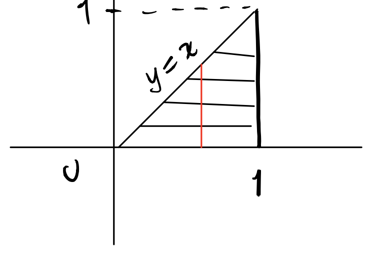
eg8 Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$

Soln: Regard $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$ as a double integral

of $\frac{\sin x}{x}$ over the region $y \le x \le 1$ and $0 \le y \le 1$

By Fubini's,

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx$$



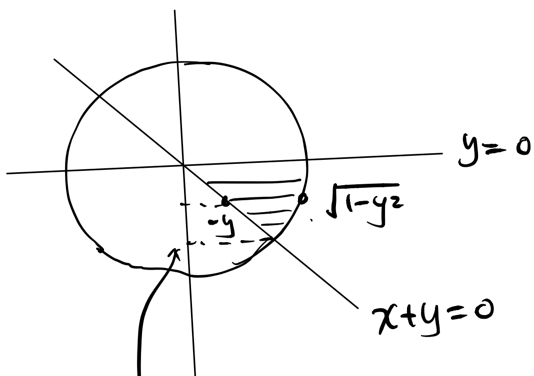
$$= \int_0^1 \left(\frac{\sin x}{x} \int_0^x dy \right) dx$$

$$= \int_0^1 \sin x \, dx = 1 - \cos 1 \quad .\#$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$
why can we use Fubini ?)

<u>eg9</u>: Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$ and the unit circle.

<u>Soln</u>: Region R is



need to solve eqt. for this value:

$$\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases} \Rightarrow y = -\frac{1}{\sqrt{2}}$$
(rejected $+\frac{1}{\sqrt{2}}$)

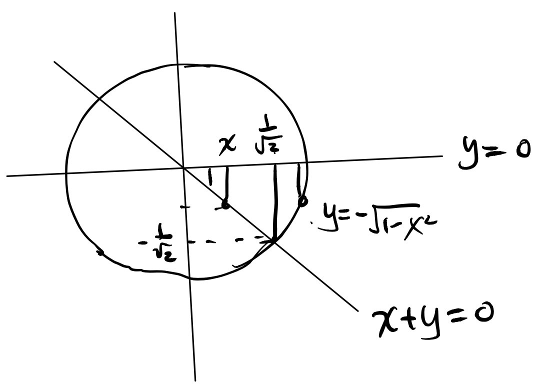
By Fubini's

$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x \, dx \, dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{0} \left(\frac{1}{2} - y^2\right) dy \quad (\text{check!})$$

$$= \frac{1}{3\sqrt{2}} \quad (\text{check!})$$

Alternately,



$$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}} \left(\int_{-x}^{0} x \, dy \right) dx$$

$$+ \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_{-\sqrt{1-x^2}}^{0} x \, dy \right) dx = \int_0^{\frac{1}{\sqrt{2}}} x^2 dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\, dx = \frac{1}{3\sqrt{2}}$$

(check!)

(More complicated in expressing the integral, but easier to do the 1st integration.)