Caution: Not all function are integrable over a (closed) rectangle.

Remark: • To show "integrable", needs to show that for all partitions and for all points (x_k, y_k) in the sub-rectangles, the Riemann sum $S(f;P) \to$ the same number.

• To disprove "integrable", needs to find for examples:

(i) some P (with some choose of (x_k, y_k)), $S(f;P)$ doesn't exist.

or (ii) some P with different (x_k, y_k) & (x_k', y_k') such that

$$S_n(f;P) \to a$$
$$\neq$$
$$S_n'(f;P) \to b$$

for (x_k', y_k')

eg5 Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} 0, & \text{if both } x \,\&\, y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is not integrable over R.

Soln: $\forall$ subdivision (partition) P of R

$= R_1 \cup R_2 \cup \cdots \cup R_n$

One can find points $(x_k, y_k) \in R_k$, for k, such that both x_k, y_k are rational.

The corresponding Riemann sum equals

$$S_n(f;P) = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} 0 \cdot \Delta A_k = 0 \to 0 \quad \text{as } \|P\| \to 0$$

On the other hand, we can also find $(x_k', y_k') \in R_k$

such that at least one of x_k', y_k' is irrational.

The corresponding Riemann sum equals

$$S_n'(f;P) = \sum_{k=1}^{n} f(x_k', y_k') \Delta A_k = \sum_{k=1}^{n} 1 \, \Delta A_k = \text{Area}(R) = 1 \to 1$$

as $\|P\| \to 0$.

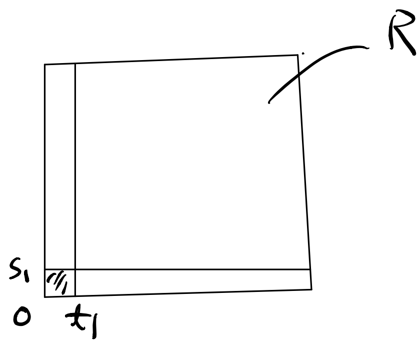
Since $S_n(f;P) \to 0 \neq 1 \leftarrow S_n'(f;P)$,

f is <u>not</u> integrable. ※

<u>eg6</u>: Let $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \frac{1}{xy}, & \text{if } x \neq 0 \text{ \& } y \neq 0 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

Then f is <u>not</u> integrable over R.



<u>Pf</u>: In any partition P of R,

there is a sub-rectangle

$R_1 = [0, t_1] \times [0, s_1]$.

Choose

$(x_1, y_1) = (t_1^2, s_1^2) \in R_1 = [0, t_1] \times [0, s_1]$

(since $0 < t_1^2 < t_1 < 1$, $0 < s_1^2 < s_1 < 1$)

Then Riemann sum

$$S_n(f;P) = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

$$\geq f(x_1, y_1) \Delta A_1 \qquad (\text{since } f \geq 0)$$

$$= \frac{1}{t_1^2 s_1^2} t_1 s_1 = \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \le \|P\| \to 0$, $t_1, s_1 \to 0$.

Hence $$S_n(f; P) \ge \frac{1}{t_1 s_1} \to \infty \quad \text{as } \|P\| \to 0$$

$\therefore$ limit doesn't exist!

Hence f is not integrable. ✱

By egs 5 & 6, we see that "condition(s)" is needed to ensure integrability of a function over (closed) rectangle.

Prop 1 Let $R = [a,b] \times [c,d]$ be a closed rectangle, and $f(x,y)$ be an integrable function over R, then f is bounded on R.

(i.e. $\exists M > 0$ s.t. "$|f(x,y)| \le M$, $\forall (x,y) \in R$")

Pf: Omitted (eg 6 above gives an idea of proof.)

Prop 2: Let $R = [a,b] \times [c,d]$ be a closed rectangle, and $f(x,y)$ be a continuous function on R, then f is integrable on R.

Pf: Omitted (See proof in 1-variable case in MATH2060 for an ideal of proof.)

(Note: both f in egs 5 & 6 are discontinuous)

Remarks: (i) Note that a continuous function on closed rectangle is always bounded (Props. 1 & 2 are consistent.)

(MATH2050 for 1-variable situation)

(ii) Prop 2 can be generalized to a <u>bounded</u> function on a closed rectangle with a "small" set of <u>discontinuity</u>.

The precise concept is "measure zero set" (MATH4050 Real Analysis)

For us, we have:

For function over <u>closed</u> <u>rectangle</u>

(a) bounded + "continuous except finitely many points"

$\Rightarrow$ integrable

(b) bounded + "continuous except finitely many differentiabe curve"

$\Rightarrow$ integrable.