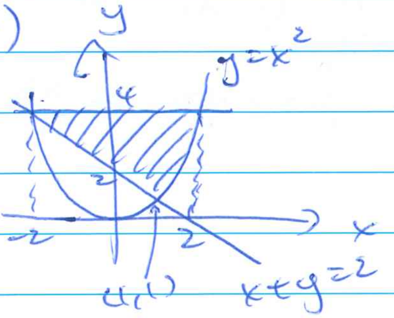
MATH 2020 HW5 solution

15.P) 14, 20, 27, 31, 34, 54

15.A) 13, 20.

15.P.14)



required area

$$= \int_1^4 \int_{2-y}^{\sqrt{y}} dx\, dy$$

$$= \int_1^4 [\sqrt{y} - (2-y)]\, dy$$

$$= \frac{37}{6}$$

15.P.20) $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\, dx\, dy$

$$= \int_0^{2\pi} \int_0^1 \ln(r^2+1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\Big[(r^2+1)\ln(r^2+1) - (r^2+1)\Big]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(2\ln 2 - 1)\, d\theta$$

$$= \pi(2\ln 2 - 1)$$

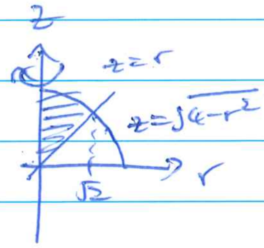
15.P.27) required volume

$$= \int_{-\pi/2}^{\pi/2} \int_{-\cos y}^{0} (-2x)\, dx\, dy$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 y\, dy$$

$$= \frac{\pi}{2}$$

15.P.31) a) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\, dz\, dx\, dy$



b) $\displaystyle\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} 3\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

c) We evaluate in spherical coord.

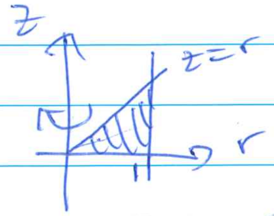
$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 3 \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi\, d\varphi \int_0^2 \rho^2\, d\rho$$

$$= 3\,(2\pi)\left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{3}\, 2^3$$

$$= 8\pi(2 - \sqrt{2})$$

15.P.34) The domain of integration is the intersection of the first octant and the region of revolution below.



a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y)\, dz\, dy\, dx$

b) $\int_0^{\pi/2} \int_0^1 \int_0^r (6+4r\sin\theta)\, r\, dz\, dr\, d\theta$

c) $\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc\varphi} (6+4\rho\sin\varphi\sin\theta)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

d) We evaluate in cylindrical coord.

$$\int_0^{\pi/2} \int_0^1 \int_0^r (6+4r\sin\theta)\, r\, dz\, dr\, d\theta$$
$$= \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3\sin\theta)\, dr\, d\theta$$
$$= \int_0^{\pi/2} (2+\sin\theta)\, d\theta$$
$$= \pi + 1$$

15.P.54) Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, which is symmetric.

Then $ax^2+2bxy+cy^2 = [x\ \ y]\, A \begin{bmatrix} x \\ y \end{bmatrix}$.

By linear algebra, A is diagonalizable by an orthogonal matrix Q, ie

$$A = Q \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix} Q^t, \quad QQ^t = Q^tQ = I.$$

Let $\begin{bmatrix} s \\ t \end{bmatrix} = Q^t \begin{bmatrix} x \\ y \end{bmatrix}$.

Then $[x\ \ y]\, A \begin{bmatrix} x \\ y \end{bmatrix} = [s\ \ t] \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \lambda s^2 + \mu t^2$

$$\int_0^\infty \int_0^\infty e^{-(ax^2+bxy+cy^2)}\, dx\, dy$$
$$= \int_0^\infty \int_0^\infty e^{-\lambda s^2 - \mu t^2}\, |\det Q|\, ds\, dt \quad \text{if } \lambda>0, \mu>0$$
$$= \frac{1}{\sqrt{\lambda\mu}} \int_0^\infty \int_0^\infty e^{-u^2-v^2}\, du\, dv, \quad (u=\sqrt{\lambda}\, s,\ v=\sqrt{\mu}\, t)$$
$$= \frac{1}{\sqrt{\lambda\mu}} \int_0^{2\pi} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$
$$= \frac{1}{\sqrt{\lambda\mu}} \int_0^{2\pi} \frac{1}{2}\, d\theta$$
$$= \frac{\pi}{\sqrt{\lambda\mu}}.$$

(cont with 15 P. 54)

$\therefore$ For the integral to be 1, one needs

$$\lambda > 0, \ \mu > 0, \ \sqrt{\lambda\mu} = \pi$$

Note that $\lambda\mu = \det A$.

$$\therefore \ \det A = \pi^2.$$

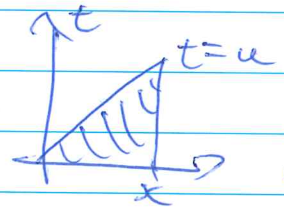
By linear algebra, for a real symmetric matrix A with positive determinant to have positive eigenvalues, it suffices $\operatorname{tr} A > 0$.

$\therefore$ The desired conditions are

$$\begin{cases} a + c > 0 \\ ac - b^2 = \pi^2 \end{cases}.$$

(15.A.13)

$$\int_0^x \int_0^u e^{m(x-t)} f(t)\,dt\,du$$
$$= \int_0^x \int_t^x e^{m(x-t)} f(t)\,du\,dt$$
$$= \int_0^x (x-t)\, e^{m(x-t)} f(t)\,dt$$



Similarly,

$$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\,dt\,du\,dv$$
$$= \int_0^x \int \int_u^x e^{m(x-t)} f(t)\,dv\,du\,dt$$
$$= \int_0^x \int_t^x (x-u)\, e^{m(x-t)} f(t)\,du\,dt$$
$$= \int_0^x \left[\frac{(x-u)^2}{2} \right]_{u=t}^{u=x} e^{m(x-t)} f(t)\,dt$$
$$= \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\,dt.$$

(15.A.25)

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} \frac{\partial^2}{\partial x \partial y} F(x,y)\,dx\,dy$$
$$= \int_{y_0}^{y_1} \left(\frac{\partial}{\partial y} F(x_1, y) - \frac{\partial}{\partial y} F(x_0, y) \right) dy$$

(if fundamental theorem of calculus is applicable, eg. if $\frac{\partial^2}{\partial x \partial y} F$ is cts)

$$= (F(x_1, y_1) - F(x_1, y_0)) - (F(x_0, y_1) - F(x_0, y_0))$$
$$= F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) - F(x_0, y_0)$$