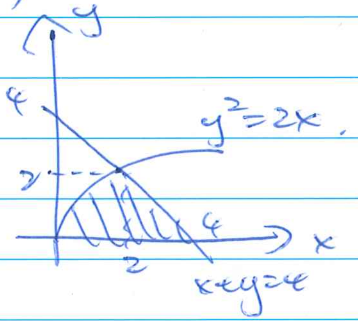
MATH 2020 HW 4 sol.

(15.6) 3, 8, 10, 12, 21

(15.8) 10, 14, 15, 18, 22

(15.6.3)



total mass

$= \int_0^2 \int_{y^2/2}^{4-y} dx\,dy$

$= \int_0^2 (4-y-y^2/2)\,dy$

$= 14/3$.

x-coord of centroid

$= \frac{3}{14} \int_0^2 \int_{y^2/2}^{4-y} x\,dx\,dy$

$= \frac{3}{14} \cdot \frac{1}{2} \int_0^2 [(4-y)^2 - y^4/4]\,dy$

$= \frac{64}{35}$

y-coord of centroid

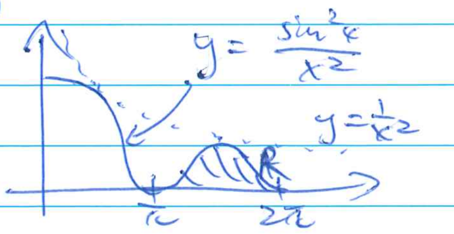
$= \frac{3}{14} \int_0^2 \int_{y^2/2}^{4-y} y\,dx\,dy$

$= \frac{3}{14} \int_0^2 y(4-y-y^2/2)\,dy$

$= \frac{5}{7}$

$\therefore$ centroid $= \left(\frac{64}{35}, \frac{5}{7}\right)$.

(15.6.8)



required moment of inertia

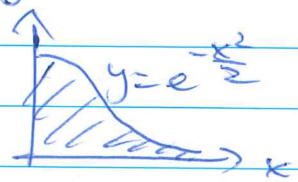
$= \iint_R x^2\,dA$

$= \int_\pi^{2\pi} \int_0^{\frac{\sin^2 x}{x^2}} x^2\,dy\,dx$.

$= \int_\pi^{2\pi} \sin^2 x\,dx$.

$= \pi/2$

15.6.10)



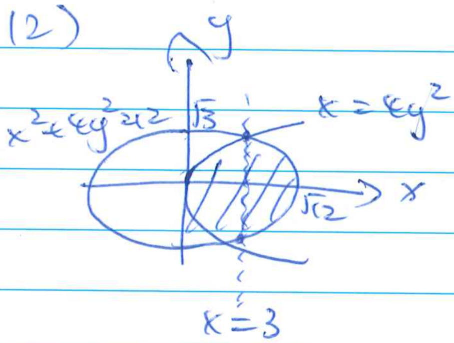
required moment

$$= \int_0^{\infty} \int_0^{e^{-x^2/2}} x \, dy \, dx$$

$$= \int_0^{\infty} e^{-x^2/2} x \, dx$$

$$= 1$$

15.6.12)



mass

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{4y^2}^{\sqrt{12-4y^2}} 5x \, dx \, dy$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{5}{2}\left(12-4y^2-16y^4\right) dy$$

$$= 23\sqrt{3}.$$

15.6.21)

$$I_x = \int_0^c \int_0^b \int_0^a (y^2+z^2) \, dx \, dy \, dz$$

$$= a \int_0^c \int_0^b (y^2+z^2) \, dy \, dz$$

$$= a \int_0^c \left(\tfrac{1}{3}b^3 + z^2 b\right) dz$$

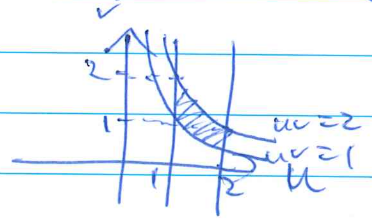
$$= \tfrac{1}{3} a (b^3 c + b c^3).$$

By symmetry, $I_y = \frac{1}{3} b (a^3 c + a c^3)$

$I_z = \frac{1}{3} c (a^3 b + a b^3)$.

15.8.10) a)

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 1 & 0 \\ v & u \end{bmatrix} = u$$



b)

$$\int_1^2 \int_1^2 \frac{y}{x} \, dy \, dx$$

$$= \int_1^2 \frac{1}{2}(4-1)\frac{1}{x} \, dx$$

$$= \frac{3\log 2}{2}$$

$$\int_1^2 \int_1^2 \frac{y}{x} \, dy \, dx$$

$$= \int_1^2 \int_{\frac{1}{u}}^{\frac{2}{u}} \frac{uv}{u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv \, du$$

$$= \int_1^2 \int_{1/u}^{2/u} uv \, dv \, du$$

$$= \frac{1}{2} \int_1^2 u\left[\left(\frac{2}{u}\right)^2 - \left(\frac{1}{u}\right)^2\right] du$$

$$= \frac{3\log 2}{2}$$

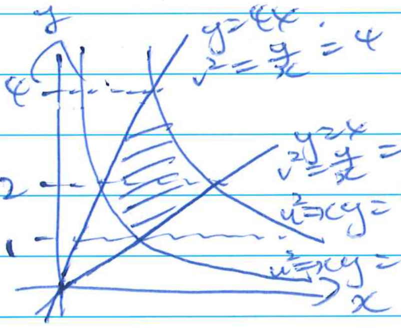
15.8.14. For $\begin{cases} x = u + \frac{1}{2}v \\ y = v \end{cases}$

$$\begin{cases} 0 \le y \le 2 \\ \frac{y}{2} \le x \le \frac{y+4}{2} \end{cases} \iff \begin{cases} 0 \le v \le 2 \\ 0 \le u \le 2 \end{cases},$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = 1$$

$$\therefore \int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} \, dx \, dy$$
$$= \int_0^2 \int_0^2 v^3 (2u) e^{(2u)^2} (1) \, du \, dv$$
$$= 2 \int_0^2 v^3 \, dv \int_0^2 u e^{4u^2} \, du$$
$$= 2(4)[(e^{16} - 1)/8]$$
$$= e^{16} - 1$$

15.8.15



Under $\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$, the domain of integration becomes $\begin{cases} 1 \le u \le 2 \\ 1 \le v \le 2 \end{cases}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 1/v & -u/v^2 \\ v & u \end{bmatrix} = 2u/v$$

$$\therefore \int_1^2 \int_{1/y}^{y} (x^2+y^2) \, dx \, dy + \int_2^4 \int_{y/4}^{4/y} (x^2+y^2) \, dx \, dy$$
$$= \int_1^2 \int_1^2 \left[\left(\frac{u}{v}\right)^2 + (uv)^2 \right] (2u/v) \, du \, dv$$
$$= 2 \int_1^2 u^3 \, du \int_1^2 \left(\frac{1}{v^3} + v \right) dv$$
$$= 2 \left(\frac{15}{4} \right) \left(\frac{15}{8} \right)$$
$$= \frac{225}{16}$$

15.8.18) a) $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$= \det \begin{bmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= u\cos^2 v + u\sin^2 v$$

$$= u$$

b) $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$= \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= 3$$

15.8.22) required vol

$$= \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1} dx\,dy\,dz$$

$$= \iiint_{u^2+v^2+w^2 \leq 1} abc\,du\,dv\,dw$$

$$= \frac{4}{3}\pi abc$$