Lecture 8
Resource Allocation Problem

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Agenda

1. The simplest model
2. DP formulation
3. Numerical example
4. Remarks
5. More complex models

The simplest model

- You are given X units of a resource and told that this resource must be distributed among N activities.
- You are also given N data tables $r_i(x)$ (for $i = 1, \cdots, N$ and $x = 0, 1, \cdots, X$) representing the return realized from an allocation of x units of resource to activity i. Further, assume that $r_i(x)$ is a nondecreasing function of x.
- The problem is to allocate all of the X units of resource to the activities so as to maximize the total return, i.e. to choose N nonnegative integers x_i, $i = 1, \cdots, N$, that maximize

$$\sum_{i=1}^{N} r_i(x_i)$$

subject to the constraint

$$\sum_{i=1}^{N} x_i = X. \tag{1}$$

Dynamic programming formulation

(i) OPTIMAL VALUE FUNCTION:

$$S_k(x) = \text{the maximum return obtainable from activities } k \text{ through } N, \text{ given } x \text{ units of resource remaining to be allocated} .$$

(ii) RECURRENCE RELATION:

$$S_k(x) = \max_{j=0,1,\ldots,x} \{r_k(j)+S_{k+1}(x-j)\}, \quad 0 \le x \le X, 1 \le k \le N.$$

(iii) BOUNDARY CONDITIONS:

$S_N(x) = r_N(x)$, for $x = 1, 2, \ldots, X$ and $S_k(0) = 0$, for $k = 1, 2, \ldots, N$.

(v) ANSWER TO BE SOUGHT: $S_1(X)$.

Numerical example

Suppose we have four doctors that are to be sent to three different hospitals to help inject the H5N1 vaccine to patients there. Table 1 (left) shows that number of patients the doctors can serve per hour. How should we allocate the doctors so that we can serve the maximum number of patients?

	Hospital		
No. of Doctors	1	2	3
0	0	0	0
1	4	2	5
2	6	4	7
3	9	7	8
4	10	11	9

x	$S_3(x)$
0	0
1	5
2	7
3	8
4	9

Table: Number of patients served (left) and boundary conditions for $S_k(x)$ (right)

Numerical example

x	j	0	1	2	3	4	
	$r_2(j)$	0	2	4	7	11	$S_2(x)$
0	$S_3(x-j)$	0					
	$+$	$\boxed{0}$					0
1	$S_3(x-j)$	5	0				
	$+$	$\boxed{5}$	2				5
2	$S_3(x-j)$	7	5	0			
	$+$	$\boxed{7}$	$\boxed{7}$	4			7
3	$S_3(x-j)$	8	7	5	0		
	$+$	8	$\boxed{9}$	$\boxed{9}$	7		9
4	$S_3(x-j)$	9	8	7	5	0	
	$+$	9	10	11	$\boxed{12}$	11	12

Table: Computation for $S_2(x)$

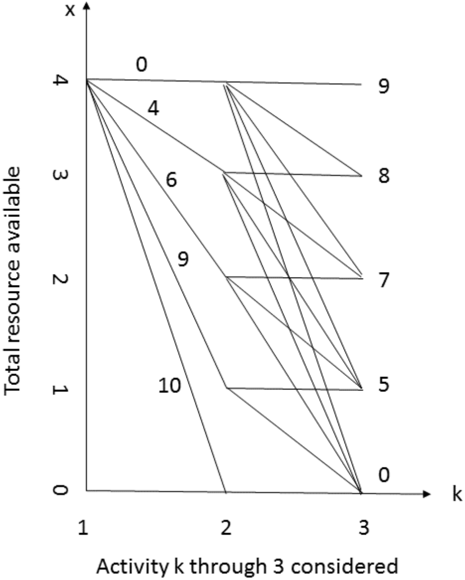
Numerical example

x	j	0	1	2	3	4	$S_1(x)$
	$r_1(j)$	0	4	6	9	10	
4	$S_2(x-j)$	12	9	7	5	0	
	+	12	13	13	$\boxed{14}$	10	14

Table: Computation for $S_1(4)$

Longest-path problem

This problem can be represented as a longest-path problem as shown below for $N = 3$ and $X = 4$.



Computational efficiency

$$S_k(x) = \max_{j=0,1,\ldots,x} \{r_k(j) + S_{k+1}(x-j)\}, \quad 0 \le x \le X, 1 \le k \le N.$$

The computational procedure requires

$$(N-2)\left(\frac{(X+1)(X+2)}{2}\right) + (X+1) \qquad \text{additions}$$

and an almost equal number of comparisons.

Example 1

Suppose the return function $r_k(x_k)$ are not necessarily nondecreasing functions of x_k and that (1) is replaced by the inequality $\sum_{i=1}^{N} x_i \leq X$. How does this affect the dynamic programming solution?

Example 2

Suppose that there are eight activities and each has the same return data. Give a *doubling-up* procedure. You need only one stage and one state variable if you use the fact that the optimal return for a k-activity problem depends only on the total amount allocated and not explicitly on the initial and terminal number of units of resource. Write the dynamic programming formulation and use the procedure to solve numerically the problem:

$$r(0) = 0, \quad r(1) = 2, \quad r(2) = 5, \quad r(3) = 9, \quad r(4) = 12,$$

$$r(5) = 14, \quad r(6) = 16, \quad r(7) = r(8) = r(9) = r(10) = 17$$

Example 3

Suppose there are two types of resources to be allocated to N activities with X units of resource-type 1 and Y units of resource-type 2. The return data for a set of N functions $r_k(x_k, y_k)$ giving the return from an allocation of x_k units of resource-type 1 and y_k units of resource-type 2 to activity k. Write the dynamic programming formulation and determine the approximate number of additions as a function of X, Y and N.

Example 4

Consider a problem slightly more complex than Example 3. Suppose that each unit of resource-type 1 costs \$2 and each unit of resource-type 2 costs \$3. You are given $Z = 2X + 3Y$ dollars. Please find out how many units of resource 1 and how many units of resource 2 should we buy so that, if we then allocate them optimally to our N activities, we can maximize the total return.