Lecture 7
Equipment Replacement Problem

MATH3220 Operations Research and Logistics
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The simplest model
Regeneration point approach
More complex equipment replacement problem

Agenda

1. **The simplest model**
2. **Regeneration point approach**
3. **More complex equipment replacement problem**

The Simplest Model

- **Problem**
  Our basic problem concerns a type of machine (perhaps an auto-mobile) which deteriorates with age, and make decisions about when to replace the incumbent machine, when to replace its replacement, etc., so as to minimize the total cost during the next N years.

- **Assumption**
  - We must own such a machine during each of the N time periods (say years).
  - y is the age of the machine when we start the process.
  - $c(i)$ is the cost of operating for one year a machine which is of age i at the start of the year.
  - p is the price of a new machine (of age 0).
  - $t(i)$ is the trade-in value received when a machine which is of age i at the start of a year is traded for a new machine at the start of the year.
  - $s(i)$ is the salvage value received for a machine that has just turned age i at the end of year N.

Dynamic Programming Model

(i) OPTIMAL VALUE FUNCTION: $S(x, k)$ is the minimum cost of owning a machine from year k through N, starting year k with a machine just turned age x, for $k = 1, 2, \ldots, N$; $x = 1, 2, \ldots, k-1, y+k-1$ when $k > 1$; and $x = y$ when $k = 1$. Here y is the age of the starting machine.

(ii) RECURRENCE RELATION:

$$S(x, k) = \text{Min}\left\{ \begin{array}{l} \text{buy} : p - t(x) + c(0) + S(1, k+1), \\ \text{keep} : c(x) + S(x+1, k+1). \end{array} \right.$$

(iii) OPTIMAL POLICY FUNCTION: $P(x, k) = B$ (buy) if buy is cheaper than keep in the recurrence relation, and $P(x, k) = K$ (keep) if otherwise.

(iv) BOUNDARY CONDITION: $S(x, N+1) = -s(x)$ for $x = 1, 2, \ldots, N$ and $y + N$.

(v) ANSWER SOUGHT: $S(y, 1)$ = the minimum cost.

Example

As an example, consider the following equipment replacement problem:

$$N = 5$$
$$y(\text{the age of the incumbent machine at the start of year 1}) = 2$$
$$c(0) = 10, c(1) = 13, c(2) = 20, c(3) = 40, c(4) = 70,$$
$$c(5) = 100, c(6) = 100;$$
$$p = 50;$$
$$t(1) = 32, t(2) = 21, t(3) = 11, t(4) = 5, t(5) = 0, t(6) = 0;$$
$$s(1) = 25, s(2) = 17, s(3) = 8, s(4) = 0, s(5) = 0, s(7) = 0.$$

Note that we do not need $s(6)$, as there is no chance that the car will be of six years old at the end of fifth year.

Example

The DP computations are summarized in the following able.

k		x = 1	2	3	4	5	6	7
6	$S(x,k)$	−25	−17	−8	0	0	–	0
5	keep	−4	12	40	70	–	100	
	buy	3	14	24	30	–	35	
	$S(x,k)$	−4K	12K	24K	30B	–	35B	
4	keep	25	44	70	–	135		
	buy	24	35	45	–	56		
	$S(x,k)$	24B	35B	45B	–	56B		
3	keep	48	65	–	126			
	buy	52	63	–	79			
	$S(x,k)$	48K	63B	–	79B			
2	keep	76	–	119				
	buy	76	–	97				
	$S(x,k)$	76KB	–	97B				
1	keep	–	117					
	buy	–	115					
	$S(x,k)$	–	115B					

We see that the minimum cost is 115; and the optimal sequence of decisions is *BKBBK* or *BBKBK*, where *B* is buy and *K* is keep.

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	buy	76	–	97				
	$S(x,k)$	76KB	–	97B				
1	keep	–	117					
	buy	–	115					
	$S(x,k)$	–	115B					

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Exercise

How many additions and how many comparisons, as a function of the duration of the process N, are required?

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$$S(x,k) = \text{Min}\begin{cases} \text{buy} : p - t(x) + c(0) + S(1, k+1), \\ \text{keep} : c(x) + S(x+1, k+1). \end{cases}$$

$$k = 1, 2, \ldots, N; x = 1, 2, \ldots, k-1, y+k-1.$$

Each evaluation of the optimal value function requires four additions and a comparison.

At the start of year N, there are N values of S required.

At the start of year $N-1$, there are $N-1$ values of S required.

$\vdots$

At the start of year 1, there are 1 value of S required.

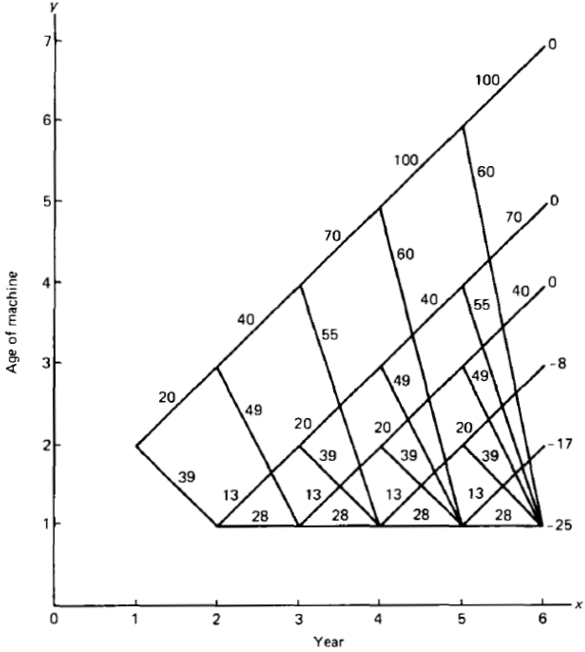
$\Rightarrow \sum_{i=1}^{N} i = N(N+1)/2$ values of S must be computed, so a total of $2N(N+1)$ additions and $N(N+1)/2$ comparisons are needed.

Shortest-path Representation of the Problem

Almost all dynamic-programming problems can be thought of as problems seeking the minimum-cost path (generally in more than two dimensions and therefore generally not easily drawn).

Letting the x axis denote the year and the y axis represent the age of the machine, we start at $(1, y)$. The "buy" decision takes us to $(2, 1)$ at an arc cost of $p - t(y) + c(0)$ and "keep" leads us to $(2, y + 1)$ with an arc cost of $c(y)$. The same reasoning applies at each point.

Shortest-path Representation of the Problem



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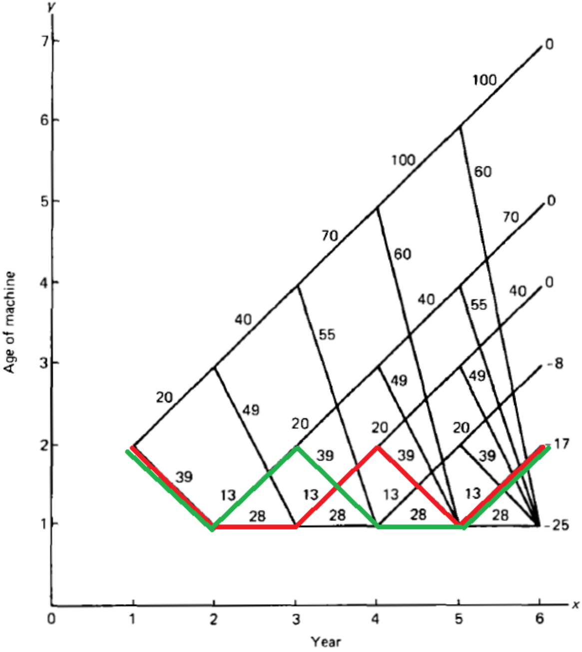
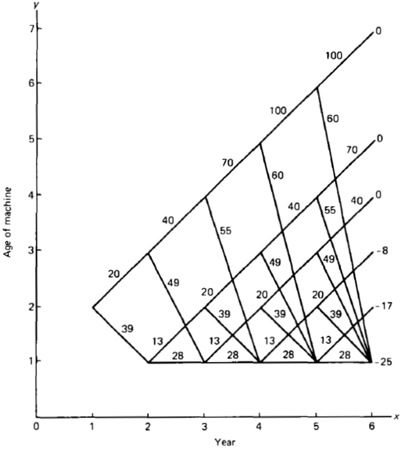


Regeneration Point Approach

Unlike many path problem, for the equipment replacement problem, we can be sure in advance that all paths, but one, eventually return at least once to a vertex on the horizontal line $y = 1$. When we return to $y = 1$ the process is said to "regenerate" itself and we can ask the question, "Given an initial situation, when shall we make our first purchase?"



The simplest model
Regeneration point approach
More complex equipment replacement problem

(i) OPTIMAL VALUE FUNCTION: $S(i)$ = the minimum attainable cost for the remaining process given we start year i with a one-year-old machine.

(ii) RECURRENCE RELATION:

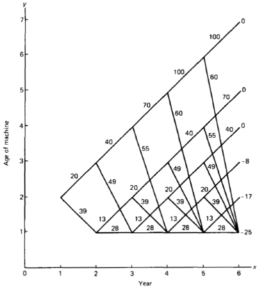
$$S(i) = \min \left[\begin{array}{l} \sum_{k=1}^{N-(i-1)} c(k) - s(N-i+2) \\ \min_{j=i,\ldots,N} \left\{ \sum_{k=1}^{j-i} c(k) + p - t(j-i+1) + c(0) + S(j+1) \right\} \end{array} \right]$$

(iii) OPTIMAL POLICY FUNCTION:
$P(i)$ = *Keep until end* if the value in the first row in the recurrence relation is less than the value in the second row, and
$P(i)$ = *Buy at the start of year* j' if the minimum value is obtained in the second row in the recurrence relation at $j = j'$.

(iv) BOUNDARY CONDITION: $S(N+1) = -s(1)$

(v) ANSWER SOUGHT: $S(1)$???

Using the procedure to solve for the data shown in the figure, we get



$$S(6) = -25;$$

$$S(5) = \min\begin{bmatrix} \underline{13 - 17} \\ 28 + S(6) \end{bmatrix} = -4,$$

$$P(5) = \text{ keep until end;}$$

$$S(4) = \min\begin{bmatrix} 13 + 20 - 8 \\ \underline{28 + S(5)}, 13 + 39 + S(6) \end{bmatrix} = 24,$$

$$P(4) = \text{ buy at the start of year 4;}$$

$$S(3) = \min\begin{bmatrix} 13 + 20 + 40 \\ 28 + S(4), \underline{13 + 39 + S(5)}, \\ 13 + 20 + 49 + S(6) \end{bmatrix} = 48,$$

$$P(3) = \text{ buy at the start of year 4;}$$

$$S(2) = \min\begin{bmatrix} 13 + 20 + 40 + 70 \\ \underline{28 + S(3)}, \underline{13 + 39 + S(4)}, \\ 13 + 20 + 49 + S(5), \\ 13 + 20 + 40 + 55 + S(6) \end{bmatrix} = 48,$$

$$P(2) = \text{ buy at the start of either year 2 or 3.}$$

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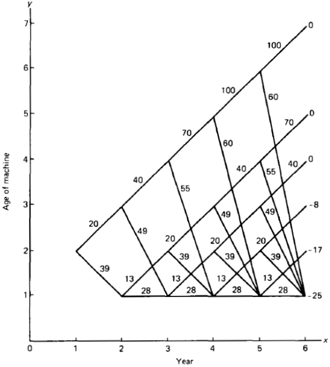

The answer is:

$$\min \left[\begin{array}{l} 20+40+70+100+100 \\ \underline{39+S(2)}, \\ 20+49+S(3), \\ 20+40+55+S(4), \\ 20+40+70+60+S(5), \\ 20+40+70+100+60+S(S) \end{array} \right] = 115$$

Decision: buy at start of year 1

More Complex Equipment Replacement Problem

- In the above equipment-replacement problem, one additional decision is available, namely, "*overhaul*".
- An overhauled machine is better than one not overhauled, but not as good as a new one.
- Let us further assume that performance depends on the *actual age of equipment* and on the *number of years since last overhaul*, but is independent of when and how often the machine was overhauled prior to its last overhaul.

The simplest model
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The known data are

$$
\begin{aligned}
k &= \text{the } k\text{th year;} \\
i &= \text{a machine's current age;} \\
j &= \text{age at last overhaul;} \\
e(k,i,j) &= \text{cost of exchanging a machine of age } i, \\
&\quad\ \text{last overhauled at age } j \text{ for a new machine} \\
&\quad\ \text{at the start of year } k; \\
c(k,i,j) &= \text{operating cost during year } k \text{ of a machine} \\
&\quad\ \text{of age } i \text{ and last overhauled at age } j; \\
o(k,i) &= \text{cost of overhauling a machine of age } i \text{ at the} \\
&\quad\ \text{beginning of year } k; \\
s(i,j) &= \text{salvage value at the end of year } N \text{ of a machine} \\
&\quad\ \text{which has just become age } i \text{ and last overhauled} \\
&\quad\ \text{at age } j.
\end{aligned}
$$

If $j = 0$, then the machine has never been overhauled.

Using the consultant's approach, we can see that, if we want to pursue an optimal replacement policy, the minimal information needed at the start of the kth year is the age of the car and how long ago it has been overhauled. Thus we have the following optimal value function:

$$f(k,i,j) \quad = \quad \text{the minimum cost during the remaining years given we start year } k \text{ with a machine of age } i \text{ and last overhauled at age } j.$$

The recurrence relation is:

$$f(k,i,j) = \min \begin{bmatrix} \text{Replace: } e(k,i,j) + c(k,0,0) + f(k+1,1,0) \\ \\ \text{Keep: } \qquad c(k,i,j) + f(k+1,i+1,j) \\ \\ \text{Overhaul: } o(k,i) + c(k,i,i) + f(k+1,i+1,i) \end{bmatrix}$$

and the boundary condition is

$$f(N+1,i,j) = -s(i,j).$$

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For $k = N$, assuming the incumbent machine is new, we must compute f for $i = 1, 2, \ldots, N - 1$ and $j = 0, 1, 2, \ldots, i - 1$.

This involves $N - 1$ evaluations of three decisions for $i = N - 1$, $N - 2$ for $i = N - 2$, ..., and 1 for $i = 1$.

$\Rightarrow$ A total of $(N - 1)N/2$ such evaluations.

For $k = N - 1$, we have $(N - 2)(N - 1)/2$ such evaluations.

For $k = N - 2$, we have $(N - 3)(N - 2)/2$ such evaluations.

$\vdots$

$\Rightarrow$ The total number is precisely

$$\sum_{i=2}^{N}(i - 1)i/2 + 1$$

or, approximately

$$\sum_{i=1}^{N} i^2/2 \approx N^3/6$$

Consequently, the total number of operations is roughly N^3 since each evaluation of the right-hand side of the recurrence relation requires a total of seven additions and comparisons.

Example 1

In the model with overhaul, suppose that a machine produces a positive net revenue $n(k, i, j)$ rather than a cost $c(k, i, j)$.

Net revenue plus salvage minus exchange and overhaul costs is maximized.

Give a dynamic programming formulation.

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Example 2

Solve Example 1 assuming that an overhaul requires 2 years, during which time the machine ages but there is no revenue.

An overhaul cannot be commenced at the start of year *N*.

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Example 3

Consider the simple equipment replacement model, suppose any machine reaching age M (M given) must be replaced and, furthermore, that one can trade-in a used machine of any age between 0 (new) and $M - 1$.
The cost of replacing an i-year-old machine by one of age j is $u(i, j)$, where $u(i, 0) = p - t(i)$ and $u(i, i)$ (the cost of keeping the current machine) equals 0.

Give the backward dynamic programming solution of the problem.

Example 4

Consider the simple equipment replacement model with the additional option ”sell your current machine if you own it and lease a new machine for the year.”

If you already have a leased machine, you can lease again or buy a new machine at cost p.

If you sell your machine but do not buy one, you get the salvage value $s(i)$.

Let l be the cost of leasing a machine for a year, excluding operating cost.

Assuming that you start year 1 with an owned, new machine, give a backward dynamic programming solution procedure.

Exercise

A company currently has a 3 year old machine. It wants to determine the optimal replacement strategy for the next 4 years.
The following data are available:

Age	$r(t)$	$c(t)$	$s(t)$
0	20000	200	-
1	19000	600	80000
2	18500	1200	60000
3	17200	1500	50000
4	15500	1700	30000
5	14000	1800	10000
6	12200	2200	5000

The company also requires that a 6 year old machine must be replaced. The cost of a new machine is 100,000 and the salvage value is the same as the trade-in value.
Solve this problem using dynamic programming to maximize the total profit.