Lecture 10
Stochastic DP Problems

MATH3220 Operations Research and Logistics
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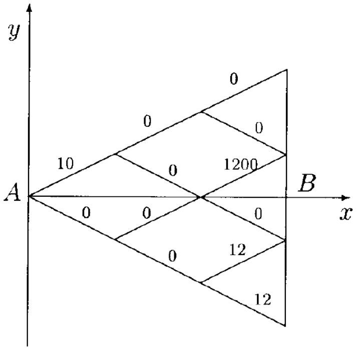
Pan Li
The Chinese University of Hong Kong

Agenda

1. A simple problem
2. Problems with Time-Lag or Delay
3. Stochastic Equipment Replacement Problem

A Simple Problem

- Imagine that we have been hired as consultant to a forgetful traveler who wishes to get from A to line B at minimum cost.
- Our problem is compounded by the fact that if we instruct the traveler to go diagonally up (or down) he remembers our advice and does so with probability $3/4$; with probability $1/4$ he forgets and does the opposite, taking the diagonally downward arc.
- The objective: minimize the expected cost of the trip.



What Constitutes A Solution?

- Deterministic DP: use DP to determine first the optimal *policy* function giving a decision for every vertex, and then deduced from it the actual optimal *sequence* of decisions for initial vertex.
- Stochastic DP: a policy and a sequence are quite different matters.
- In conformity with control engineering terminology, we shall call the solution specified by a sequence of decisions *open-loop* control and the solution specified by a policy *feedback* control.

Numerical Solutions of Our Example

The best open-loop control sequence:

Consider all eight possible sequences of three decisions each, and choose the one with minimum expected cost.

For example, for the decision sequence $D - U - D$,

$$E_{DUD} = \frac{27}{64}\cdot 0 + \frac{9}{64}(10+12+1200) + \frac{3}{64}(12+10+10) + \frac{1}{64}\cdot 1210 = 192\frac{1}{4}$$

It turns out that the decision sequence $U - U - D$ has the minimum expected cost, $120\frac{3}{16}$.

Numerical Solutions of Our Example

The Optimal feedback control:

Dynamic programming yields the optimal feedback control, as we shall see below.

the optimal expected value function:

$$S(x,y) = \text{the expected cost of the remaining process if we start at vertex } (x,y) \text{ and use an optimal feedback control policy.}$$

By the stochastic version of the principle of optimality, we have

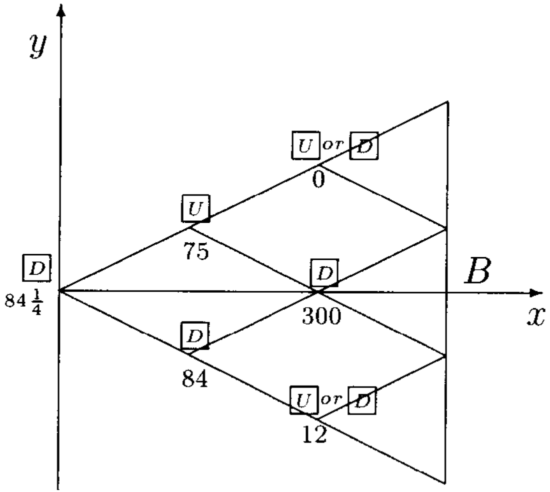
$$S(x,y) = \min \begin{bmatrix} U: \frac{3}{4}\{a_u(x,y)+S(x+1,y+1)\} + \frac{1}{4}\{a_d(x,y)+S(x+1,y-1)\} \\ D: \frac{1}{4}\{a_u(x,y)+S(x+1,y+1)\} + \frac{3}{4}\{a_d(x,y)+S(x+1,y-1)\} \end{bmatrix}.$$

The boundary condition is

$$S(3,3)=0,\ S(3,1)=0,\ S(3,-1)=0,\ S(3,-3)=0.$$

Numerical Solutions of Our Example

The Optimal feedback control:



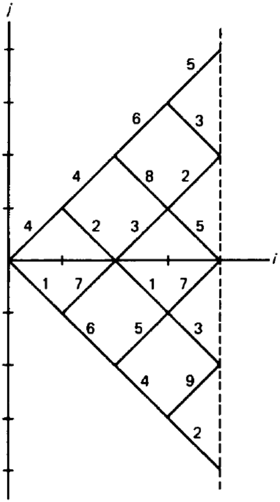
The expected cost using the optimal feedback control policy is $84\frac{1}{4}$.

Problems with Time-Lag or Delay

Consider the minimum cost stochastic path problem shown in figure, where our decision at stage k is implemented at stage $k+2$, no matter what the state at stage $k+2$.

Decision U results in a diagonally upward move when it is implemented two stages later with probability $\frac{3}{4}$ and it results in a downward move two stages later with probability $\frac{1}{4}$. Decision D is like the U decision except the probability $\frac{3}{4}$ and $\frac{1}{4}$ are interchanged.

We assume that at $(0,0), (1,1)$, and $(1,-1)$ the probability of moving diagonally upward and the probability of moving diagonally downward each equal $\frac{1}{2}$.



Stochastic Time-Lag Model

The optimal expected value function:

$S(i, j, d_1, d_2) =$ the minimum expected cost of the remaining process given that we start at (i, j), d_1 was the decision made at stage $i - 1$, and d_2 was the decision made at stage $i - 2$.

Stochastic Time-Lag Model

By the principle of optimality, writing each of the four possible sets of previous two decisions separately, we have

$$S(i,j,U,U) = \frac{3}{4}a_u(i,j) + \frac{1}{4}a_d(i,j) + \min\begin{bmatrix} \frac{3}{4}S(i+1,j+1,U,U) + \frac{1}{4}S(i+1,j-1,U,U) \\ \frac{3}{4}S(i+1,j+1,D,U) + \frac{1}{4}S(i+1,j-1,D,U) \end{bmatrix}$$

$$S(i,j,U,D) = \frac{1}{4}a_u(i,j) + \frac{3}{4}a_d(i,j) + \min\begin{bmatrix} \frac{1}{4}S(i+1,j+1,U,U) + \frac{3}{4}S(i+1,j-1,U,U) \\ \frac{1}{4}S(i+1,j+1,D,U) + \frac{3}{4}S(i+1,j-1,D,U) \end{bmatrix}$$

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$$S(i,j,D,D) = \frac{1}{4}a_u(i,j) + \frac{3}{4}a_d(i,j) + \min\begin{bmatrix} \frac{1}{4}S(i+1,j+1,U,D) + \frac{3}{4}S(i+1,j-1,U,D) \\ \frac{1}{4}S(i+1,j+1,D,D) + \frac{3}{4}S(i+1,j-1,D,D) \end{bmatrix}$$

Stochastic Time-Lag Model

For processes starting at stage 0 or 1, upward and downward transitions each occur with probability $\frac{1}{2}$, so

$$S(1,j,U,\_) = \frac{1}{2}a_u(1,j) + \frac{1}{2}a_d(1,j) + \min\left[\begin{array}{l} \frac{1}{2}S(2,j+1,U,U) + \frac{1}{2}S(2,j-1,U,U) \\ \frac{1}{2}S(2,j+1,D,U) + \frac{1}{2}S(2,j-1,D,U) \end{array}\right]$$

$$S(1,j,D,\_) = \frac{1}{2}a_u(1,j) + \frac{1}{2}a_d(1,j) + \min\left[\begin{array}{l} \frac{1}{2}S(2,j+1,U,D) + \frac{1}{2}S(2,j-1,U,D) \\ \frac{1}{2}S(2,j+1,D,D) + \frac{1}{2}S(2,j-1,D,D) \end{array}\right]$$

$$S(0,0,\_,\_) = \frac{1}{2}a_u(0,0) + \frac{1}{2}a_d(0,0) + \min\left[\begin{array}{l} \frac{1}{2}S(1,1,U,\_) + \frac{1}{2}S(1,-1,U,\_) \\ \frac{1}{2}S(1,1,D,\_) + \frac{1}{2}S(1,-1,D,\_) \end{array}\right]$$

The boundary condition, assuming the process ends when $i = 4$, is most easily written as

$$S(4,j,d_1,d_2) = 0 \quad \text{for all } j,\ d_1 \text{ and } d_2.$$

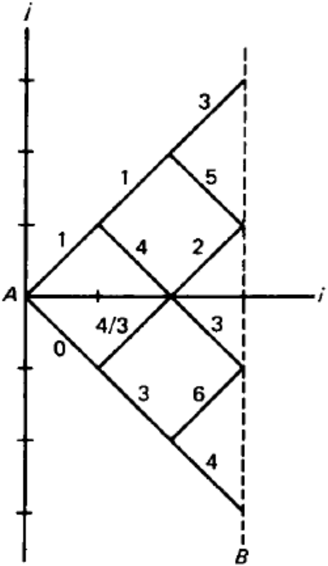
Stochastic Time-Lag Model

In this case, the value of S will be computed at stage 3 for various decisions at stage 2 where these decisions are irrelevant and not really made, but the answer will be correct. More complicated to write, but easier to use for hand computation, are formulas for S at stage 2 of the form

$$\begin{aligned} S(2,j,U,U) = \quad & \tfrac{9}{16}\left[a_u(2,j) + a_u(3,j+1)\right] + \tfrac{3}{16}\left[a_u(2,j) + a_d(3,j+1)\right] \\ & + \tfrac{3}{16}\left[a_d(2,j) + a_u(3,j-1)\right] + \tfrac{1}{16}\left[a_d(2,j) + a_d(3,j-1)\right] \end{aligned}$$

Exercise

Determine the feedback policy that minimizes the expected cost of going from A to line B in the network in the figure where the cost of a path is the sum of its arc numbers plus 1 for each change in direction, and where at each vertex there are two admissible decisions. Decision *U* diagonally up with probability $\frac{2}{3}$ and down with probability $\frac{1}{3}$ and decision *D* goes up with probability $\frac{1}{3}$ and down with probability $\frac{2}{3}$.



Stochastic Equipment Replacement Problem

- Consider a stochastic version of the equipment replacement problem, where we either keep or replace our current machine at the start of each year i, $i = 1, 2, \cdots, N$.

- We assume that the operating cost is a random variable, dependent upon the age of the machine.

- We further assume that our machine may suffer a catastrophic failure at the end of any year, and then it must be replaced by a new machine.

The data defining our problem are

N = the duration of the process,

y = the age of the machine with which we start year 1,

$n(i,j)$ = the probability that the net operating cost during the year is j, $j = 0, 1, \cdots, J$, given that the machine is of age i at the start of the year,

p = the purchase price of new machine,

$t(i)$ = the trade-in value of a machine, in working order, just turned age i,

$u(i)$ = the trade-in value of a machine, in failed condition, just turned age i,

$q(i)$ = the probability that a machine, in working order, of age i at the start of a year fails at the end of the year,

$s(i)$ = the salvage value at the start of year $N+1$ of a working machine just turned age i,

$v(i)$ = the salvage value at the start of year $N+1$ of a failed machine just turned age i.

Stochastic Equipment Replacement Model

To formulate the problem of minimizing the expected cost for the above situation, we define

$S(i,k) \quad = \quad$ the minimum expected cost of the remaining process if we start year k with a machine, in working order of age i.

Then for $k = 1, \cdots, N-1$; $i = 1, \cdots, k-1$ and $i = y + k - 1$:

$$S(i,k) = \text{Min} \left[\begin{array}{ll} B: & p - t(i) + \sum_{j=0}^{J} jn(0,j) + q(0)\{p - u(1) + S(0,k+1)\} \\ & +\{1-q(0)\}S(1,k+1) \\ K: & \sum_{j=0}^{J} jn(i,j) + q(i)\{p - u(i+1) + S(0,k+1)\} \\ & +\{1-q(i)\}S(i+1,k+1) \end{array} \right]$$

and

$$P(i,k) = \begin{cases} B & \text{if } B \le K, \\ K & \text{if } B > K. \end{cases}$$

Stochastic Equipment Replacement Model

For $i = 0$,

$$S(0,k) = \sum_{j=0}^{J} jn(0,j) + q(0)\{p - u(1) + S(0,k+1)\} + \{1-q(0)\}S(1,k+1);$$

and the boundary condition is

$$S(i,N) = \text{Min}\left[\begin{array}{l} B : p - t(i) + \sum_{j=0}^{J} jn(0,j) - q(0)v(1) - \{1-q(0)\}s(1) \\ K : \sum_{j=0}^{J} jn(i,j) - q(i)v(i+1) - \{1-q(i)\}s(i+1) \end{array}\right].$$