1. Method-1.

what's we want to get

$$\int_a^{a+T} f(x)\,dx = \underbrace{\int_0^T f(x)\,dx}_{} + \int_a^0 f(x)\,dx + \int_T^{a+T} f(x)\,dx$$ the linearity of integral.

then we just need to prove $\int_a^0 f(x)dx + \int_T^{a+T} f(x)\,dx = 0$

$\Leftrightarrow \int_T^{a+T} f(x)\,dx = -\int_a^0 f(x)\,dx = \int_0^a f(x)\,dx$ (the "$-$" change the order of integrate interval)

here we take substitution $u = x - T \Leftrightarrow x = u + T$, $dx = du$

and $x \in [T, a+T]$ implies $u \in [0, a]$

so $\int_T^{a+T} f(x)\,dx = \int_0^a f(u+T)\,du = \int_0^a f(u)\,du$ (the definition of periodic function $f(u) = f(u+T)$)

$$\Rightarrow \int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \quad \text{done.}$$

Method-2:

Regard $H(a) = \int_a^{a+T} f(x)\,dx$ is a function defined by this integral (depends on a), then:

$H'(a) = f(a+T) - f(a) = 0$ $\quad (f(a) = f(a+T))$

this implies $H(a)$ is a constant function, so:

$$\int_a^{a+T} f(x)\,dx = H(a) = H(0) = \int_0^T f(x)\,dx.$$

2. The only difference is $\sin x$ to $\cos x$. so ino order to get the desired result.

we take $x = \frac{\pi}{2} - u$ which make $\sin x$ to be $\cos u$.

$\Leftrightarrow u = \frac{\pi}{2} - x$, $x \in [0, \frac{\pi}{2}]$ means $u \in [\frac{\pi}{2}, 0]$ and $du = -dx$

$$\Rightarrow \int_0^{\frac{\pi}{2}} g(\sin x)\,dx = -\int_{\frac{\pi}{2}}^0 g(\sin(\tfrac{\pi}{2} - u))\,du = \int_0^{\frac{\pi}{2}} g(\cos u)\,du \quad \text{done.}$$

3. For when $x \in [0, \frac{\pi}{2}]$ $\cos x, \sin x$ both lie in $[0,1]$, and we have $g(\cos x)$, $g(\sin x)$,

In order to make it well-defined, the domain of g should be $[0,1]$, at least.

4. Still take $x=\frac{\pi}{2}-u$. $\Leftrightarrow$ $u=\frac{\pi}{2}-x$, $x\in[0,\pi]$ means $u\in[\frac{\pi}{2},-\frac{\pi}{2}]$, $dx=-du$.

$$\Rightarrow \int_0^{\pi} x\cdot P(\sin x)\,dx = -\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (\tfrac{\pi}{2}-u)\,P(\sin(\tfrac{\pi}{2}-u))\,du$$

$$=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P(\cos u)\,du - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\,P(\cos u)\,du.$$

the imporetant obersevations are $M(u)=u\,P(\cos u)$ is odd function while
$N(u)=P(\cos u)$ is even function.

so we know the integral of an odd function over a symmetric interval like $[-\frac{\pi}{2},\frac{\pi}{2}]$ should be 0 , and for the even function, we just need to compute half of it.

$$\Rightarrow \int_0^{\pi} x\,P(\sin x)\,dx = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}} P(\cos u)\,du - 0 = \pi\int_0^{\frac{\pi}{2}} P(\cos x)\,dx.$$

5. Just an application of $\left(\int_{g(x)}^{f(x)} u(t)\,dt\right)' = u(f(x))\,f'(x) - u(g(x))\cdot g'(x)$.

$$\Rightarrow \left(\int_1^{xe^x}\frac{\ln t}{t}\,dx\right)' = \frac{\ln(xe^x)}{xe^x}\cdot(xe^x)' - 0 = \frac{\ln x + x}{xe^x}\cdot(e^x+xe^x) = \frac{(x+\ln x)(x+1)}{x}$$

6. Take $x=\pi-t$ $\Leftrightarrow$ $t=\pi-x$, $x\in[0,\pi]$ means $t\in[\pi,0]$, $dx=-dt$.

$$A=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx = -\int_{\pi}^{0}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}\,dt = \int_0^{\pi}\frac{(\pi-t)\sin t}{1+\cos^2 t}\,dt$$

$$=\pi\underbrace{\int_0^{\pi}\frac{\sin t}{1+\cos^2 t}\,dt}_{=B} - \underbrace{\int_0^{\pi}\frac{t\sin t}{1+\cos^2 t}\,dt}_{=A}$$

$\Rightarrow A=\pi\cdot B - A \Rightarrow A=\frac{\pi}{2}\cdot B$.

And $B=\int_0^{\pi}\frac{\sin t}{1+\cos^2 t}\,dt$ take $u=\cos t$ $\Leftrightarrow$ $du=-\sin t\,dt$
$u\in[1,-1]$

$$=-\int_1^{-1}\frac{du}{1+u^2} = \int_{-1}^{1}\frac{du}{1+u^2}$$

$$=\arctan u\Big|_{-1}^{1} = \frac{\pi}{4}-(-\frac{\pi}{4}) = \frac{\pi}{2}.$$

$$\Rightarrow A=\frac{\pi}{2}\cdot B=\frac{\pi^2}{4}.$$