Topics: Quotient Rule, Chain Rule, Taylor's Thm., L'Hôpital's Rule

Things learned

- Proofs of R.T., LMVT and CMVT
- L'Hôpital's Rule (more examples)
- T.T. (examples)

Things not yet learned, but you should know

- (Quotient Rule) $(f/g)'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$, where $g(x) \neq 0$.
- (Chain Rule)
$$\underbrace{\frac{df(y(x))}{dy}}_{\star}\frac{dy(x)}{dx} = \frac{df(y(x))}{dx} \quad ({}^1)$$
- (Implicit differentiation)([2]) — Reason behind the computation of
$$f(x) = \ln(x) \implies f'(x) = \frac{1}{x}, x > 0 \ ({}^3)$$
Let $y = \ln(x)$ Assuming $x > 0$ (for simplicity, the other case can be done by yourself), then we have
  (i) $y - \ln(x) = 0$ is an equation,
  (ii) $e^y - x = 0$ ([4])
  (iii) both e^y and x are functions of x,
  (iv) differentiating everything with respect to x on both sides of the $\boxed{\text{`=' sign}}$ gives
$$\frac{de^y}{dx} - \frac{dx}{dx} = \frac{d0}{dx} = 0 \implies \frac{de^y}{dx} = 1$$
$$\implies \frac{de^y}{dy}\cdot\frac{dy}{dx} = 1 \implies e^y\cdot\frac{dy}{dx} = 1 \implies y' = \frac{1}{e^y} = \frac{1}{x}$$

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[1] In the derivative labelled as ($\star$), I wrote $y(x)$ in the expression '$df(y(x))$' to emphasize the fact that y is actually a function of the variable x. I could have just written y, if the reader understands this dependence on x !

[2] Explained to math1010b, but not yet to math1010a

[3] Assumed for simplicity. If one consider also the $x < 0$ case, one gets $\frac{d\ln|x|}{dx} = \frac{1}{x}$.

[4] If you like, you can write as in footnote (1), $y(x)$ instead of y to indicate that y is a function of x!

Assignments

1. (Proof of Quotient Rule) Let $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ be two functions, $\boxed{\text{both}}$ differentiable at $x=c$ in (a,b) and $g(c)>0$ ([5]). This exercise will guide you to show (using First Principle) that

$$(f/g)'(c)=\frac{g(c)\cdot f'(c)-f(c)\cdot g'(c)}{g^2(c)} \tag{1.1}$$

   (a) Write down the difference quotient.

   (b) Just as in question 1 last week, one can insert suitable terms to arrive at

$$\frac{(f(c+h)-f(c))g(c)+f(c)(g(c)-g(c+h))}{g(c+h)g(c)h}$$

   What term do you need to insert to arrive at the above line?

   (c) Letting $h\to 0$ (reads: 'h goes to zero'), one obtains the answer. Where is '$g(c)>0$' used?

2. ('Rough' Proof of Chain Rule) Let f,g be two diff' functions, whose domains we do not specify yet. Suppose

   - $g(x)$ is diff' at $x=c$,
   - $f(y)$ is diff' at $y=g(c)$, then
   - $f(g(x))$ is diff' at $x=c$. Moreover, the following 'pseudo-cancelation law' holds:

$$\left.\frac{df(y)}{dx}\right|_{x=c} ({}^6) = \left.\frac{df}{dy}\right|_{y=g(c)} \left.\frac{dy}{dx}\right|_{x=c}$$

   Answer the following questions (which lead to 'rough proof' of Chain Rule)

   (a) Write down the difference quotient of $\left.\frac{df(y)}{dx}\right|_{x=c}$.

   (b) Insert the term $\frac{g(c+h)-g(c)}{g(c+h)-g(c)}$, rewrite the difference quotient as

$$\left.\frac{\Delta f(y)}{\Delta y}\right|_{y=g(c)} \left.\frac{\Delta y}{\Delta x}\right|_{x=c}$$

   Write down what we described in this item in detail (i.e. in the form starting

$$f(g(c+h))-f(g(c))\cdots)$$

[5] This is done for simplicity

[6] Recall that f is a function of y, but we are now differentiating with respect to x, not y!

(c) Let $h \to 0$ in the difference quotient will lead to the proof.
Question. Try to finish the proof yourself, noting especially where 'continuity of g is used'.

(d) The proof above is a 'rough proof' because there are cases for which this proof does not seem to work. Point out one possible problem with this proof.

3. Find mistake(s) (if any) in the following application of L'Hôpital's Rule:

   Let $f(x) = x^2 \sin(1/x), g(x) = x$, then the limit $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ type, but
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} \text{ has no limit.}$$

4. Write down the error term, i.e. E_n, of the following application of Taylor's Theorem to the function $(1+x)^a$, where a is any real number.
$$(1+x)^a = \underbrace{1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots + \frac{a(a-1)(a-2)\cdots(a-(n-1))}{n!}x^n}_{TP_n} + E_n$$

   Comment:
   TP_n stands for 'Taylor polynomial of degree n'.