Further Techniques of Integration (Cont'd)

Keywords:
t-Substitution, trigo. integration, Riemann Sum

Assignments Some Methods of Integration not covered:

- t-substitution (see below)
- Some form of trigonometric integrals (see below)
- How to use Riemann Sum to evaluate definite integrals

1. For integrals of the form

$$I := \int \frac{p(\theta)}{q(\theta)} d\theta$$

where $p(\theta)$ and $q(\theta)$ are polynomials in θ, one can 'transform' the integral I into an integral of 'rational function' in t, where t is defined by

$$t := \tan(\theta/2).$$

(Question) on rewriting functions of θ as functions of the new variable t!

(a) Let $t = \tan(\theta/2)$, using the formulas

$$\sin(2\phi) = 2\sin(\phi)\cos(\phi), \qquad (1.1)$$

$$\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi) \qquad (1.2)$$

or otherwise, rewrite

$$\sin(\theta) \text{ repectively } \cos(\theta)$$

in terms of t

(b) Rewrite $d\theta$ in terms of t and dt

(c) Rewrite the integrand (i.e. the function to be integrated!) of

$$\int \frac{\sin(\theta)}{3\cos(\theta) - \sin(\theta)} d\theta$$

as a rational function of t.

2. In many science disciplines, one needs to compute integrals of the form

$$\int_a^b f(x)\cos(nx)dx,$$

this exercise is about two such integrals (which can be done using 'Reduction Formula' before).

Evaluate the following integrals:

- $\int \cos^3 x \cos^2 x dx$ (Hint: Use integration by parts)
- $\int \cos^2 x \sin^4 x dx$ (Hint: Use the one of the double-angle formulas in (1.1) or (1.2))

(a) Show that $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0$, if m, n are non-zero natural numbers.

(b) Show that $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0$, if m, n are non-zero natural numbers and $m \neq n$.

(c) Show that $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$, if m, n are non-zero natural numbers and $m \neq n$.

3. (Riemann Sum) Let $0 < a < b$ and $x_i = a + i \cdot (b-a)/n$. Show that

$$s_n := \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1}) \tag{1.3}$$

goes to $(b^3 - a^3)/3$ as $n \to \infty$
(Hint: Use the formula $\sum_{i=1}^{\infty} i^2 = \frac{n(n+1)(2n+1)}{6}$.) Comment: This question shows that the limit of the Riemann sum (1.3) is $(b^3 - a^3)/3$, i.e.

$$\int_a^b x^2 dx = (b^3 - a^3)/3.$$

This method relies on the 'sum of square' formula, hence is complicated. There is a faster and neater method by Fermat, using sum of geometric series.