1. (a) $t=\tan\frac{\theta}{2}$

then: $\sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2} = \dfrac{2\,\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{1}{\cos^2\frac{\theta}{2}}} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{2t}{1+t^2}$ $\left(\frac{1}{\cos^2\frac{\theta}{2}} = \sec^2\frac{\theta}{2} = 1+\tan^2\frac{\theta}{2}\right)$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \dfrac{(1-\tan^2\frac{\theta}{2})}{\sec^2\frac{\theta}{2}} = \dfrac{1-t^2}{1+t^2}$

(b) $t=\tan\frac{\theta}{2}$

$dt = \sec^2\frac{\theta}{2}\cdot\frac{1}{2}\,d\theta = (1+t^2)\cdot\frac{1}{2}\cdot d\theta$

$d\theta = \frac{2}{1+t^2}\,dt.$

(c) $\displaystyle\int \frac{\sin\theta}{3\cos\theta-\sin\theta}\,d\theta$ $\quad t=\tan\frac{\theta}{2}.$

$= \displaystyle\int \frac{\frac{2t}{1+t^2}}{3\cdot\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\cdot \frac{2}{1+t^2}\,dt$

$= \displaystyle\int \frac{2t}{3-3t^2-2t}\cdot\frac{2}{1+t^2}\,dt = \cancel{\int \frac{t+t^3}{3-3t^2-2t}\,dt}$ $\displaystyle\int \frac{4t}{(1+t^2)(3-3t^2-2t)}\,dt$

2. $\displaystyle\int \cos^3x\cdot\cos^2x\,dx = \int \cos^4x\,d\sin x$

$= \displaystyle\int (1-\sin^2x)^2\,d\sin x.$

$= \displaystyle\int (1-2\sin^2x+\sin^4x)\,d\sin x$

$= \sin x - \frac{2}{3}\sin^3x + \frac{1}{5}\sin^5x + C$

$\displaystyle\int \cos^2x\cdot\sin^4x\,dx = \int \cos^2x\cdot\sin^2x\cdot\sin^2x\,dx$

$= \displaystyle\int \frac{1}{4}\sin^2 2x\cdot\frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{8}\displaystyle\int \sin^2 2x\,dx - \frac{1}{8}\int \sin^2 2x\cdot\cos 2x\,dx$

$= \frac{1}{16}\displaystyle\int (1-\cos 4x)\,dx - \frac{1}{16}\int \sin^2 2x\,d\sin 2x$

$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$

(a) $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$ $\qquad \sin(mx)\cdot\cos(nx) = \frac{1}{2}[\sin(mx+nx)+\sin(mx-nx)]$

$= \frac{1}{n}\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,d\sin(nx)$

$= \frac{1}{n}\sin(mx)\cdot\sin(nx)\Big|_{-\pi}^{\pi} - \frac{m}{n}\displaystyle\int_{-\pi}^{\pi}\sin(nx)\cdot\cos(mx)\,dx = -\frac{m}{n}\int_{-\pi}^{\pi}\sin(nx)\cdot\cos(mx)\,dx$

$= \frac{1}{2}\left[\displaystyle\int_{-\pi}^{\pi}\sin(m+n)x\,dx + \int_{-\pi}^{\pi}\sin(m-n)x\,dx\right]$ $\quad (m\neq n)$ (if $m=n$ consider $\sin mx\cdot\cos(mx) = \frac{1}{2}\sin(2mx)$)

$= \frac{1}{2}\left[\frac{1}{m+n}(-\cos(m+n)x)\Big|_{-\pi}^{\pi} + \frac{1}{m-n}[-\cos(m-n)x]\Big|_{-\pi}^{\pi}\right] = 0$

(b) $\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx \qquad \cos mx \cdot \cos nx = \frac{1}{2}[\cos(mx+nx) + \cos(mx-nx)]$

$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x] dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x \, dx$

$= \frac{1}{2}\frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2}\frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi}$

$= 0$

(c) it's the same with (b) just use $\sin(mx)\cdot\sin(nx) = \frac{1}{2}[\cos(mx-nx) - \cos(mx+nx)]$

3. Sol: $x_i = a + \frac{b-a}{n}\cdot i \quad i = 1, \cdots n$

so $x_i - x_{i-1} = \frac{1}{n}(b-a)$

$\Rightarrow S_n = \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1}) = \frac{b-a}{n}\sum_{i=1}^{n}(a + \frac{b-a}{n} i)^2$

$= \frac{b-a}{n}\sum_{i=1}^{n}(a^2 + \frac{2(b-a)\cdot a}{n} i + \frac{(b-a)^2}{n^2}\cdot i^2)$

$= \frac{b-a}{n}\cdot na^2 + \frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n} i + \frac{(b-a)^3}{n^3}\sum_{i=1}^{n} i^2 \qquad \sum_{i=1}^{n} i = \frac{(1+n)\cdot n}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$

$= (b-a)\cdot a^2 + \frac{2a(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$

so $\lim_{n\to+\infty} S_n = (b-a)\cdot a^2 + a(b-a)^2 + \frac{1}{3}(b-a)^3$

$= ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3}(b^3 - 3b^2a + 3ba^2 - a^3)$

$= \frac{1}{3}(b^3 - a^3)$

Actually this is $\int_a^b x^2 dx = \frac{1}{3}x^3\Big|_a^b = \frac{1}{3}b^3 - \frac{1}{3}a^3$.