e.g. A manager of a company, determines that t months after initiating an advertisting compaign, the number of products will be sold is estimated by

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5 \quad \text{(thousand)} \quad , \; t \geq 0.$$

a) Find $P'(t)$ and $P''(t)$.

b) At what time will sales be maximized? What is the maximum level of sales?

c) The manager plans to terminate the advertising compaign when the sales rate is minimized. When does it occur?

a) Direct computation:

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5$$

$$P'(t) = -\frac{3}{(t+2)^2} + \frac{24}{(t+2)^3} = \frac{18-3t}{(t+2)^3}$$

$$P''(t) = \frac{6}{(t+2)^3} - \frac{72}{(t+2)^4} = \frac{6t-60}{(t+2)^4}$$

(b) Solve $P'(t) > 0$ $\qquad\qquad$ $P'(t) < 0$

$$\frac{18-3t}{(t+2)^3} > 0 \qquad\qquad \frac{18-3t}{(t+2)^3} < 0$$

$$18-3t > 0 \qquad (\because t \geq 0,\ t+2>0) \qquad 18-3t < 0$$

$$t < 6 \qquad\qquad t > 6$$

($P(t)$ is strictly increasing when $t<6$ and strictly decreasing when $t>6$,

$P(t)$ is continuous at $t=6$.)

$\therefore$ $P(t)$ attains maximum when $t=6$. (By 1st derivative check.)

Maximum sales level $= P(6) = \dfrac{83}{16}$

OR : (By observation, $P(t)$ can be differentiated infinitely many times, so if $P(t)$ attains maximum / minimum at $t = t_0$, we must have $P'(t_0) = 0$, that's why we consider the equation $P'(t) = 0$.)

$$P'(t) = 0$$

$$\frac{18-3t}{(t+2)^3} = 0$$

$$t = 6$$

(At this moment, we only know $(6, P(6))$ is a stationary point .)

$$P''(6) = -\frac{24}{8^4} < 0$$

$\therefore$ $P(t)$ attains maximum when $t = 6$. (By 2nd derivative check.)

Maximum sales level $= P(6) = \frac{83}{16}$

(c) (In fact, we want to minimize $P'(t)$ now!

We apply 1st derivative check to $P'(t)$, i.e. look at $P''(t)$.)

Solve $P''(t) > 0$ $\qquad$ $P''(t) < 0$

$$\frac{6t-60}{(t+2)^4} > 0 \qquad \frac{6t-60}{(t+2)^4} < 0$$

$$6t-60 > 0 \qquad 6t-60 < 0$$

$$t > 10 \qquad t < 10$$

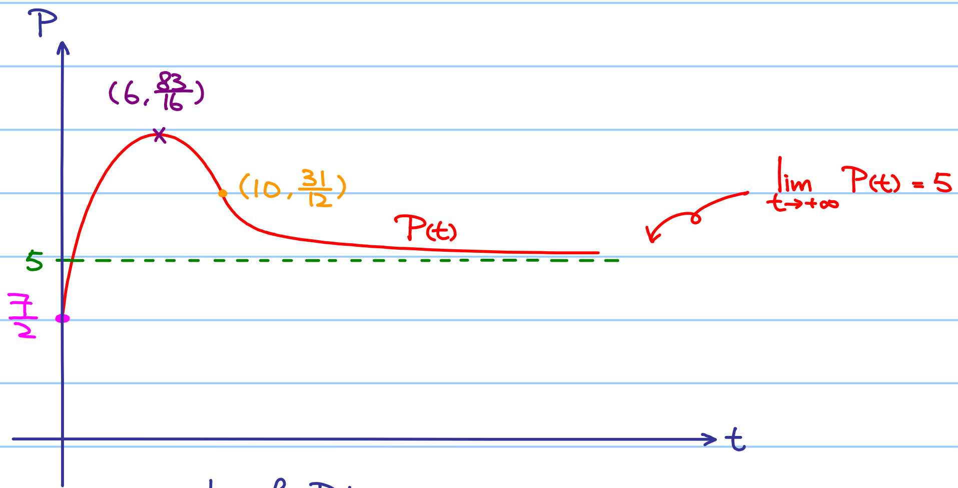
$\therefore$ $P'(t)$ attains minimum when $t = 10$. (By 1st derivative check.)

(Note: $(10, P(10))$ is a point of inflection.)

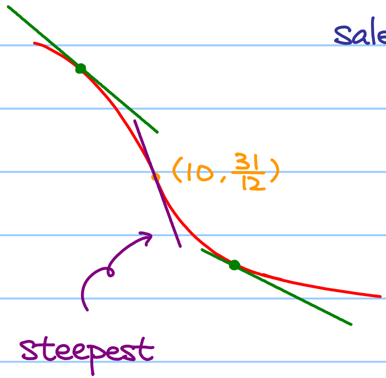
OR: $P'''(t) = -\frac{18}{(t+2)^4} + \frac{288}{(t+2)^5} = \frac{252-18t}{(t+2)^5}$

$$P'''(10) = \frac{72}{12^5} > 0$$

$\therefore$ $P'(t)$ attains minimum when $t = 10$. (By 2nd derivative check.)



graph of P(t)

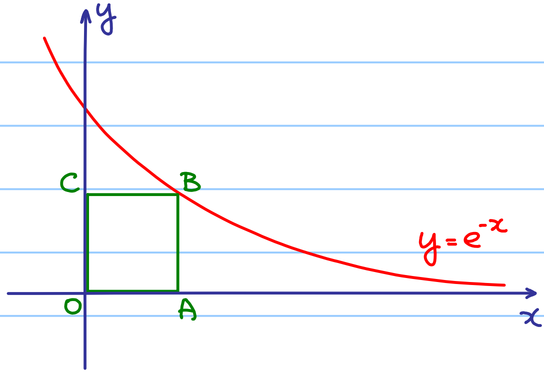


Sales rate at t $= P'(t)$

$= $ slope of the tangent line at $(t, P(t))$

Meaning of minimizing $P'(t)$ in part (c).

e.g. OABC is a rectangle inscribed in the region bounded by the positive coordinate axes and the curve $y=e^{-x}$. Find the maximum area of the rectangle.

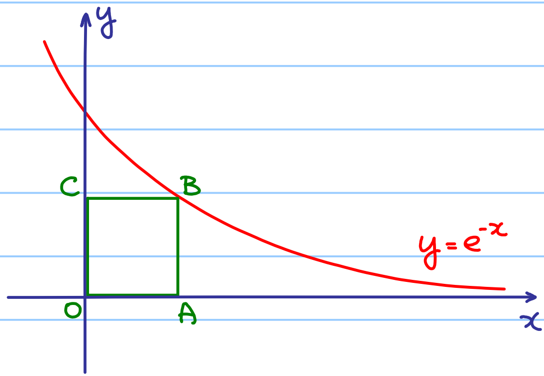


Maximize a function!

Dependent variable : ?

Independent variable : ?

e.g. OABC is a rectangle inscribed in the region bounded by the positive coordinate axes and the curve $y = e^{-x}$. Find the maximum area of the rectangle.



Maximize a function !

Dependent variable : Area of OABC, A

Independent variable : x

Area of OABC = OA × AB

$A = xe^{-x} \qquad x \geq 0$

$$\frac{dA}{dx} = e^{-x} - xe^{-x}$$

$$= e^{-x}(1-x)$$

$$\frac{dA}{dx} > 0 \qquad\qquad \frac{dA}{dx} < 0$$

$$e^{-x}(1-x) > 0 \qquad\qquad e^{-x}(1-x) < 0$$

$$1-x > 0 \qquad\qquad 1-x < 0$$

$$1 > x \qquad\qquad 1 < x$$

$\therefore$ A attains maximum when $x = 1$.

Maximum area of OABC $= A(1) = 1 \cdot e^{-1} = e^{-1}$

Remark : Most Important issue :

1) identifying dependent and independent variable

2) setting up an equation between them

Relative Rates

Suppose x and y are variables related by an equation, but both of them can further be regarded as functions of a third variable t.

(i.e. $x(t)$ and $y(t)$)

(Often : t = time)

Then Implicit differentiation helps to give a relation between $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

e.g. Relation of pollution and population of fish.

Level of populant $= x$ parts per million (ppm)

Number of fish $= F$

Given $F = \dfrac{32000}{3+\sqrt{x}}$

When there are 4000 fish left in the lake,

the population is increasing at the rate of 1.4 ppm/year.

At what rate is the fish population changing at this time?

e.g. Relation of pollution and population of fish.

Level of populant = x parts per million (ppm)

Number of fish = F

Given $F = \dfrac{32000}{3+\sqrt{x}}$

When there are 4000 fish left in the lake,

the population is increasing at the rate of 1.4 ppm / year.

At what rate is the fish population changing at this time?

time : t (years)

$F = 4000$

$\dfrac{dx}{dt} = 1.4$ (increasing, $\dfrac{dx}{dt} > 0$; decreasing, $\dfrac{dx}{dt} < 0$)

$\dfrac{dF}{dt} = ?$ when $\dfrac{dx}{dt} = 1.4$, $F = 4000$

Idea : Apply implicit differentiation to the equation

$F = \dfrac{32000}{3+\sqrt{x}}$ and differentiate with respect to t

$$\frac{dF}{dt} = \frac{d}{dt}\left(\frac{32000}{3+\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{32000}{3+\sqrt{x}}\right)\frac{dx}{dt}$$ (Apply chain rule)

$$\frac{dF}{dt} = \frac{-16000}{\sqrt{x}\,(3+\sqrt{x})^2}\,\frac{dx}{dt}$$

$\dfrac{dx}{dt} = 1.4$ Oops , $x = ?$

Recall: $F = \dfrac{32000}{3+\sqrt{x}}$, when $x = 4000$

$$4000 = \frac{32000}{3+\sqrt{x}}$$

$$x = 25$$

$$\frac{dF}{dt} = \frac{-16000}{\sqrt{x}(3+\sqrt{x})^2}\frac{dx}{dt} = \frac{-16000}{\sqrt{25}(3+\sqrt{25})^2} \times 1.4 = -70 \text{ (fish per year)}$$

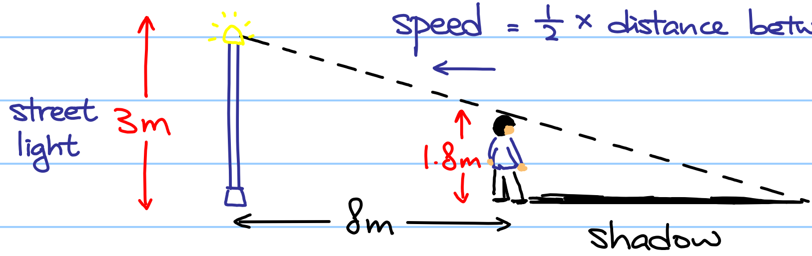
Note: Reasonable!

$\frac{dx}{dt} = 1.4 > 0$, i.e. pollutant is increasing.

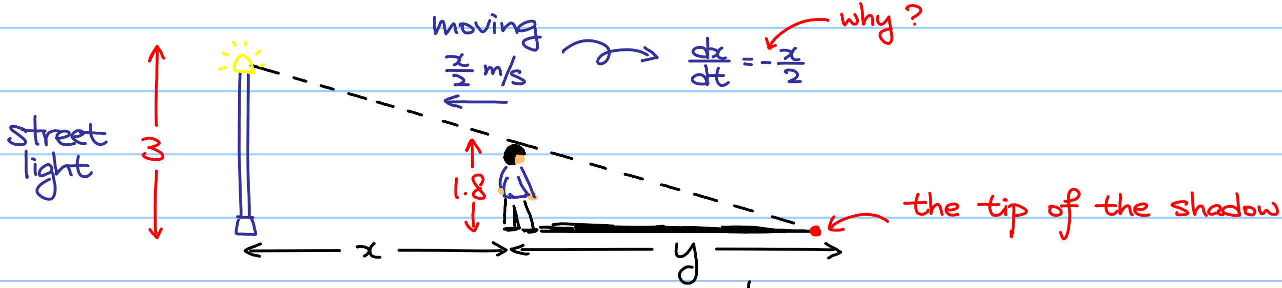
$\frac{dF}{dt} = -70 < 0$, i.e. population of fish is decreasing.

e.g.

speed = $\frac{1}{2}$ × distance between the street light and him.



rate of change of the shadow when he is 8m away from the street light = ?



$\frac{dy}{dt} = ?$ when $x = 8$.

Setting up an equation relating x and y.

$$\frac{1.8}{x} = \frac{3}{x+y}$$

$$1.8x = 1.2y$$

$$3x = 2y$$

differentiate both sides with respect to t.

$$3\frac{dx}{dt} = 2\frac{dy}{dt}$$

$$3\left(-\frac{x}{2}\right) = 2\frac{dy}{dt}$$

When $x = 8$, $\frac{dy}{dt} = -6$.

Furthermore :

What is the speed of the tip of the shadow ?

Ans : $\frac{d(x+y)}{dt}$!

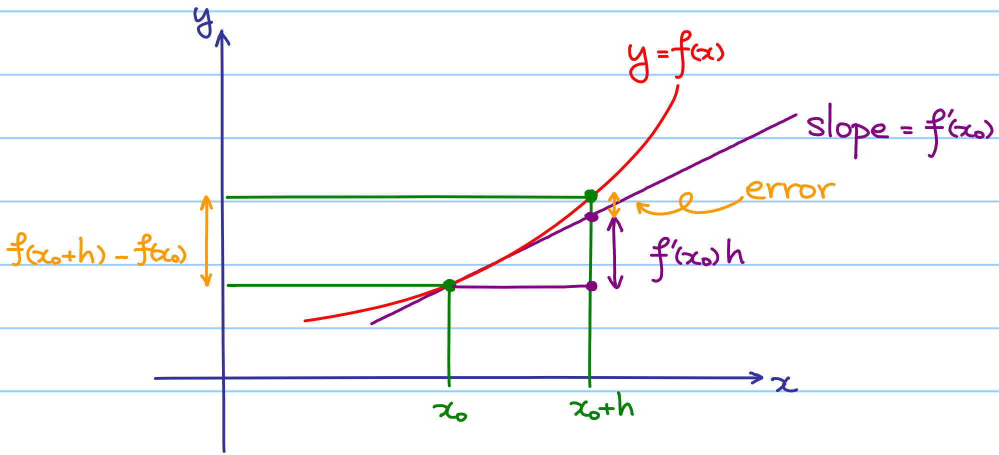
Marginal Analysis :

Idea : $y = f(x)$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$\approx \frac{f(x_0+h) - f(x_0)}{h}$$ if $h>0$, but h is small.

$$f(x_0+h) - f(x_0) \approx f'(x_0)\,h$$



e.g. $N(t)$ = GDP of a country, t years after 2015.

$$= t^2 + 4t + 200 \quad \text{(billion dollars)}$$

$$N'(t) = 2t + 4$$

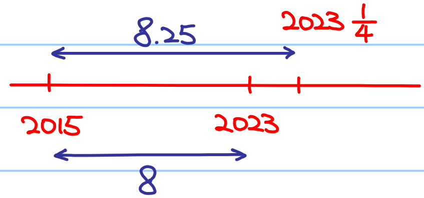
change of GDP during the first quarter of 2023

$$= N(8.25) - N(8)$$

$$\approx N'(8) \times 0.25$$

$$= 20 \times 0.25$$

$$= 5$$



$$N(8.25) - N(8) = 5.0625$$

Open Question :

1) Is it a good approximation ? Why ?
2) Why do we use this approximation ?

Indefinite Integral :

Antiderivative : A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

e.g. If $f(x)=2x$, $F(x)=x^2$,

then we have $F'(x)=f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x)=x^2+C$, where C is a constant.

Then, we still have $F'(x)=f(x)$.

Therefore, antiderivative of a function $f(x)$ is NOT unique.

That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,

what is the relation between them?

Natural question : If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,

what is the relation between them ?

Answer : $F(x)$ and $G(x)$ differ by a constant.

proof : Suppose $F'(x) = G'(x) = f(x)$

Let $H(x) = F(x) - G(x)$

Then $H'(x) = F'(x) - G'(x) = 0$

$\therefore$ $H(x)$ is a constant function, i.e. $H(x) = C$ for some constant C.

i.e. $F(x) = G(x) + C$

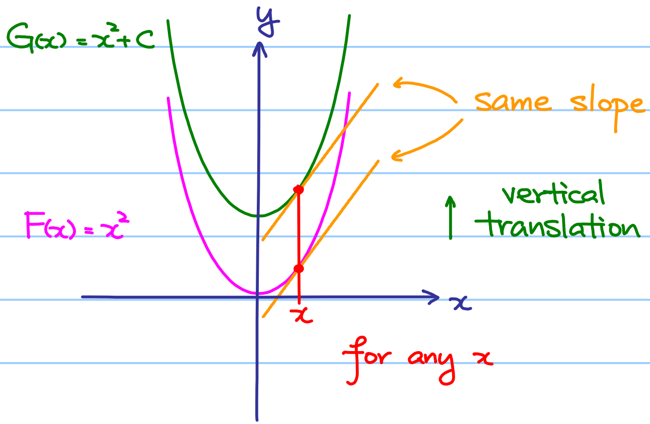
Therefore, antiderivative of a function $f(x)$ is NOT unique,

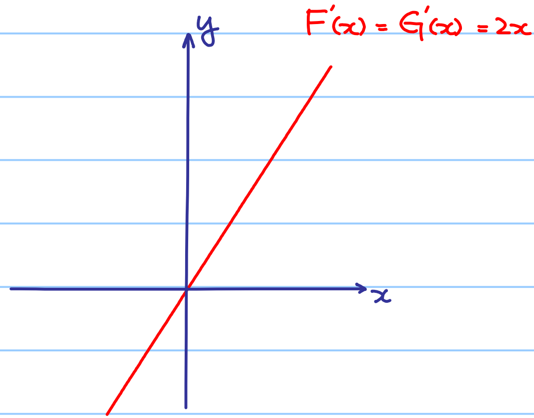
but it is unique up to a constant.

e.g. If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.





If $F(x)$ is an antiderivative of $f(x)$, we write

integrand

$$\int f(x)\, dx = F(x) + C$$

integral symbol variable of integration

e.g. $\int 2x\, dx = x^2 + C$

If $F(x)$ is an antiderivative of $f(x)$,

$$F(x) \xrightarrow{\text{differentiate}} f(x) \xrightarrow{\text{integrate}} \int f(x)\,dx = F(x) + C$$

$$= \text{original function up to a constant}$$

Note: When we write $\int f(x)\,dx$, sometimes it may be regarded as a class of functions.

Rules for Integrating Common Functions

1) $\int k\,dx = kx + C$, for constant k.

Note : $\frac{d}{dx}(kx+C) = k$

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all n except -1.

Note : $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = x^n$

3) $\int \frac{1}{x}\,dx = \ln|x| + C$ (Interesting when $x<0$)

Note: $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$

4) $\int e^x\,dx = e^x + C$

Note: $\frac{d}{dx}(e^x + C) = e^x$