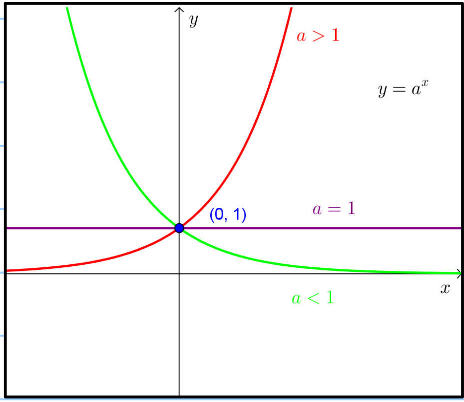
Quick Review on Exponental Function and Logarithmic Functions :

- $y = a^x$ with $a > 0$

Note : $y = a^x$ is well-defined when $a > 0$!

Think: If $a = -1$, when $x = \frac{1}{2}$, $y = a^x = \sqrt{-1}$!



graph of $y = a^x$ for

1) $a > 1$ 2) $a = 1$ 3) $0 < a < 1$

Properties :

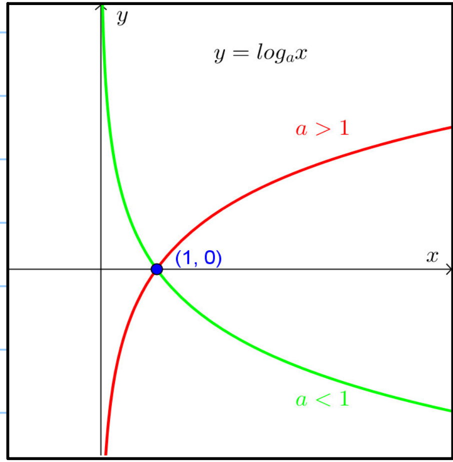
1) a^x is positive for any $a > 0$ and any real number x.

2) If $a > 1$, $\lim_{x \to -\infty} a^x = 0$

$1 > a > 0$, $\lim_{x \to +\infty} a^x = 0$

- $y = \log_a x$ with $a > 1$ or $0 < a < 1$

Note : $y = \log_a x$ is well-defined when $a > 1$ or $0 < a < 1$!



graph of $y = \log_a x$ for

1) $a > 1$ 2) $0 < a < 1$

Properties :

1) By definition, if $y = a^x$, then $\log_a y = x$

$\Downarrow$

$y > 0$

independent variable

dependent variable

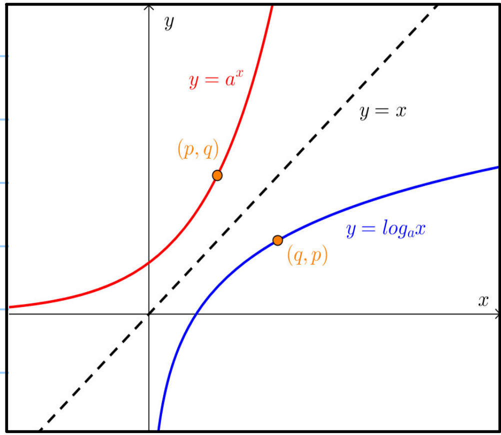
$y = \log_a x \Rightarrow x > 0$

2) The graphs of $y = a^x$ and $y = \log_a x$ are symmetric along $y = x$.

It suffices to show if (p, q) lies on the graph of $y = a^x$, then (q, p) lies on the graph of $y = \log_a x$ and vice versa.

But it follows from the definition directly:

$$q = a^p \iff \log_a q = p$$



Then situation when $a > 1$

(How about $0 < a < 1$?)

Note: reflecting $y = 1^x = 1$ along $y = x$ is $x = 1$, which is NOT a function.

It suggest why $\log_a x$ is NOT defined if $a = 1$.

Differentiation of Exponential Functions :

Recall :

$\frac{d}{dx} e^x = e^x$

In particular, we write $\ln x$ instead of $\log_e x$, which is called natural logarithm.

Ex: Apply chain rule to show that $\frac{d}{dx} e^{ax} = a e^{ax}$.

Now, we write $a^x = e^{\ln a^x} = e^{(\ln a)x}$.

So $\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = (\ln a) e^{(\ln a)x} = a^x \ln a$.

Note: If $a>1$, $\frac{d}{dx} a^x = a^x \ln a > 0$ for all x, so $y=a^x$ is strictly increasing.

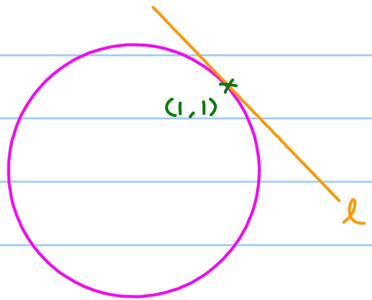
If $0<a<1$, $\frac{d}{dx} a^x = a^x \ln a < 0$ for all x, so $y=a^x$ is strictly decreasing.

Implicit Differentiation

e.g. $x^2+y^2=2$ — $\mathcal{C}$

Locus of $\mathcal{C}$ is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.



We want to find the equation of the tangent line ℓ

(i.e. need to know the slope of ℓ)

Note: $x^2+y^2=2$ is NOT a function.

Question: How to find $\frac{dy}{dx}$? (and, actually, is it defined?)

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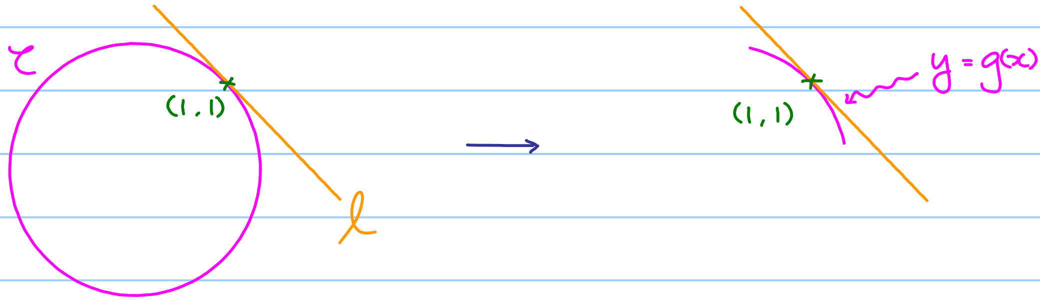
Answer : Yes , roughly speaking ,



The small segment of $\mathcal{C}$ containing (1, 1) can be regarded as the graph of some function $y = g(x)$. (In fact , $y = \sqrt{2 - x^2}$ in this case .)

How to find? Do it as usual!

e.g. $x^2+y^2=2$

differentiate both sides with respect to x.

$$2x+\frac{d}{dx}y^2=0$$

$$2x+2y\frac{dy}{dx}=0 \qquad \text{(Applying chain rule)}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

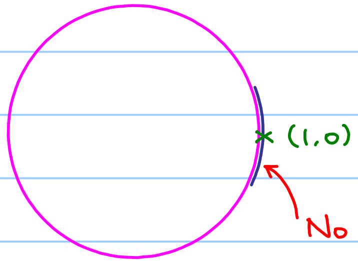
$\therefore \frac{dy}{dx}=-1$ when $(x,y)=(1,1)$.

We denote it by $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-1$

Remark :

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

$\therefore \frac{dy}{dx}$ is NOT defined when $(x,y)=(1,0)$ or $(-1,0)$.



No matter how small the arc is,

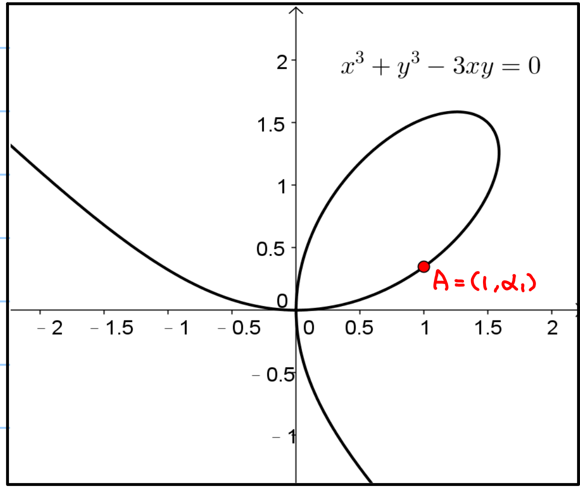
it cannot be realized as graph of some function $y=g(x)$.

e.g. $x^3 + y^3 - 3xy = 0$ —— $\mathcal{C}$

$$3x^2 + 3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

If we want to find the slope of the tangent line at A.



putting $x=1$ into $\mathcal{C}$.

$y^3-3y+1=0$

NOT easy to solve!

FACT: The above equation has three roots, two roots α_1, α_2 are positive $(\alpha_1 < \alpha_2)$

one root is negative.

$A=(1,\alpha_1)$ and what we need is $\left.\frac{dy}{dx}\right|_{(x,y)=(1,\alpha_1)}$

Applications :

e.g. Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x . $e^y \frac{dy}{dx} = 1$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$.

Ex : By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

e.g. Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

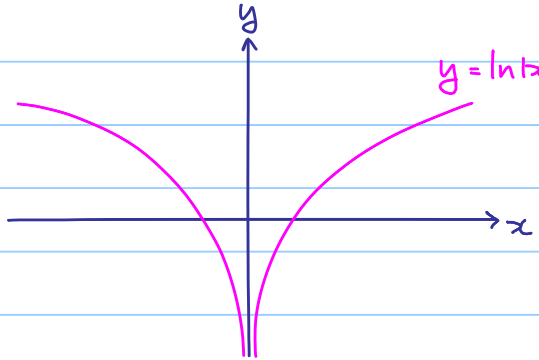
We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

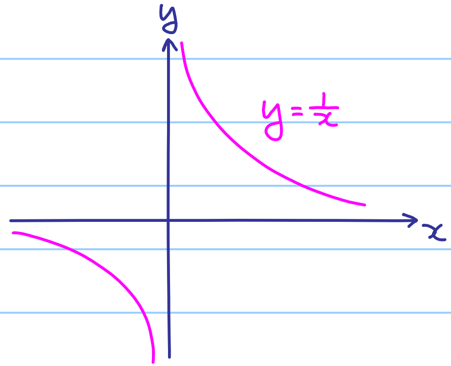
For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

For $x < 0$,

$$y = \ln(-x)$$
$$e^y = -x$$
$$e^y \frac{dy}{dx} = -1$$
$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$





Note: It is why $\int \frac{1}{x}\,dx = \ln|x| + c$.

putting absolute sign here.

e.g. If $y = \sqrt[3]{\dfrac{(x-1)(x-2)^2}{x-4}}$, then find $\dfrac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule .

$$y^3 = \frac{(x-1)(x-2)^2}{x-4}$$

$$\ln y^3 = \ln \frac{(x-1)(x-2)^2}{x-4}$$

$$3\ln y = \ln(x-1) + 2\ln(x-2) - \ln(x-4)$$

$$\frac{3}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}$$ (Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right) = \frac{1}{3}\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right)$$

Application of Differentiation :

e.g. Bacteria in closed environment with nutrition

What happens :

- Number of bacteria increases as plenty of nutrition at the beginning.
- Number of bacteria decreases as nutrition is no longer sufficient to support a large number of bacteria.

Suppose the number of bacteria t hours after the start of the experiment is modeled by the function

$$N(t) = \frac{10t+5}{e^{t+1}} \text{ (thousand)} \quad , \ t \geq 0 .$$

① Number of bacteria at the beginning

$= N(0) = 5/e \approx 1.84$ (thousand)

② $N'(t) = \dfrac{10e^{t+1} - (10t+5)e^{t+1}}{e^{2(t+1)}}$

$= \dfrac{-10t+5}{e^{t+1}}$

$N'(t) > 0 \iff -10t+5 > 0 \iff t < 0.5$

$N'(t) < 0 \iff -10t+5 < 0 \iff t > 0.5$

1st derivative check $\Rightarrow$ $N(t)$ attains max. when $t = 0.5$

Max. number of bacteria $= N(0.5) = 10/e^{1.5} \approx 2.23$ (thousand)

③ $\lim_{t \to +\infty} N(t) = \lim_{t \to +\infty} \frac{1}{e} \frac{10t+5}{e^t}$

$= 0$

Recall: If $p(x)$ is a polynomial,

then $\lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$.

i.e. Extinct eventually!

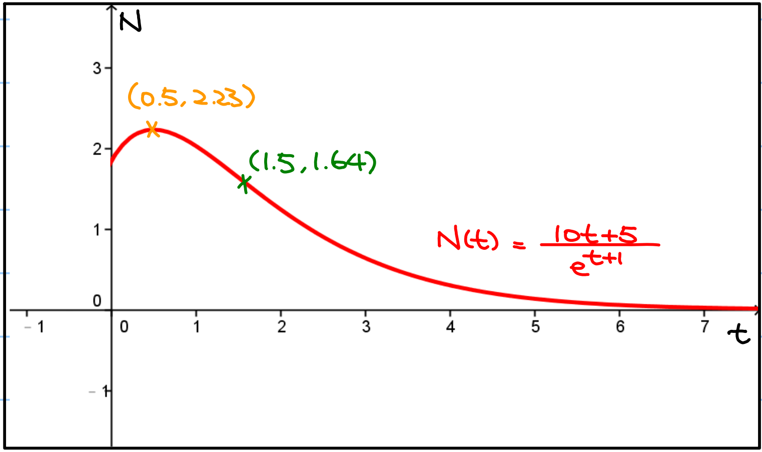
④ $N''(t) = \dfrac{-10e^{t+1} - (-10t+5)e^{t+1}}{e^{2(t+1)}}$

$= \dfrac{10t-15}{e^{t+1}}$

$N''(t) > 0 \iff 10t - 15 > 0 \iff t > 1.5$

$N''(t) < 0 \iff 10t - 15 < 0 \iff t < 1.5$

$\therefore$ point of inflection $= (1.5, N(1.5)) \approx (1.5, 1.64)$



On the other hand, $N'(t)$ attains min when $t = 1.5$

$N'(1.5) = -10/e^{1.5} = -2.23$

i.e. Number of bacteria decreases most rapidly at $t = 1.5$ and decreasing rate at $t = 1.5$ is 2.23 thousand / hour.