Elementary Rules of Differentiation :

If $f(x)$ and $g(x)$ are differentiable functions , then

① $(f+g)'(x) = f'(x) + g'(x)$

② $(f-g)'(x) = f'(x) - g'(x)$

③ [product rule] $(f \cdot g)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

④ [quotient rule] $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$

proof of (3):

$$\lim_{\Delta x \to 0} \frac{(f \cdot g)(x+\Delta x) - (f \cdot g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x+\Delta x) + f(x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x)$$

g is diff.

$\Rightarrow g$ is cont.

$\Rightarrow \lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$

e.g. If $f(x)$ is differentiable, then $k \cdot f(x)$ is differentiable.

and $(kf)'(x) = k \cdot f'(x)$ (or write $\frac{d}{dx} kf(x) = k\frac{df}{dx}$).

Idea: Let $g(x) = k$, then $g'(x) = 0$.

Apply product rule, the result follows.

e.g. Find $\frac{d}{dx}(3x^2+7x-2)$

e.g. Find $\frac{d}{dx}(3x^2+7x-2)$

$$\frac{d}{dx}(3x^2+7x-2) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$$ Apply ① and ②

$$= 3\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(2)$$

$$= 3(2x) + 7(1) - 0$$

$$= 6x + 7$$

e.g. Find the derivative of the function $(3x^2-5x+1)(2x+7)$

e.g. Find the derivative of the function $(3x^2-5x+1)(2x+7)$

$\frac{d}{dx}[(3x^2-5x+1)(2x+7)]$

$= [\frac{d}{dx}(3x^2-5x+1)](2x+7) + (3x^2-5x+1)[\frac{d}{dx}(2x+7)]$ Apply ③ product rule

$= (6x-5)(2x+7) + (3x^2-5x+1)(2)$

$= 18x^2+22x-33$

Ex: Try to compare: Expand $(3x^2-5x+1)(2x+7)$ and get $6x^3+11x^2-33x+7$

Then differentiate, get the same result?

e.g. Find the derivative of the function $\dfrac{2x}{x^2+1}$.

e.g. Find the derivative of the function $\frac{2x}{x^2+1}$.

$$\frac{d}{dx}\frac{2x}{x^2+1} = \frac{\left[\frac{d}{dx}(2x)\right](x^2+1) - (2x)\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

e.g. Find $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)$$

$$= -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

$$= -\frac{1}{2x\sqrt{x}} + \frac{1}{2\sqrt{x}}$$

Derivative of e^x :

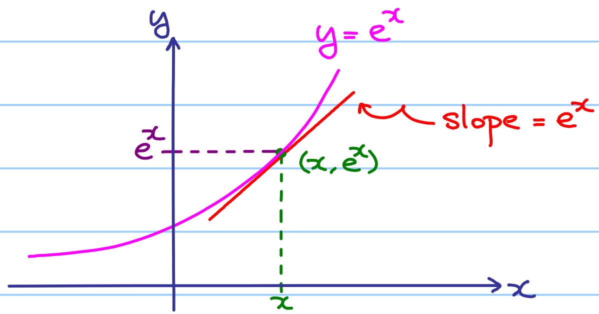
Recall : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Cheating : $\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$= e^x$ (getting back itself)

Geometrical meaning :



e.g. Find $\frac{d}{dx}[e^x(3x^2+7x-2)]$

$$\frac{d}{dx}[e^x(3x^2+7x-2)] = [\frac{d}{dx}e^x](3x^2+7x-2) + e^x[\frac{d}{dx}(3x^2+7x-2)]$$

$$= e^x(3x^2+7x-2) + e^x(6x+7)$$

$$= e^x(3x^2+13x+5)$$

Question: How to differentiate a more complicated function, such as $\sqrt{x^2+3x}$?

We need a tool called chain rule.

Chain Rule :

If $f(x)$ and $g(x)$ are differentiable function, then the composite function $(f \circ g)(x) = f(g(x))$ is also differentiable and

$$(f \circ g)'(x) = f'(g(x))\, g'(x) .$$

Hard to understand? Let's rewrite :

Let $u = g(x)$, $y = f(u) = f(g(x))$, then

Chain rule : $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Think as : $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

e.g. Find the derivative of $\sqrt{x^2+3x}$.

Let $u = g(x) = x^2+3x$,

$y = f(u) = \sqrt{u}$

then $f(g(x)) = \sqrt{x^2+3x}$

$\frac{du}{dx} = 2x+3$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \cdot (2x+3)$

$= \frac{1}{2\sqrt{x^2+3x}} \cdot (2x+3)$ put $u = x^2+3x$ back

$f'(g(x))$ $g'(x)$

differentiate f
then put back $g(x)$

e.g. Find the derivative of $(3x^2-2x)^{2015}$.

Let $u = g(x) = 3x^2-2x$ $\qquad \frac{du}{dx} = 6x-2$

$y = f(u) = u^{2015}$ $\qquad \frac{dy}{du} = 2015u^{2014}$

then $f(g(x)) = (3x^2-2x)^{2015}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 2015u^{2014} \cdot (6x-2)$

$= 2015(3x^2-2x)^{2014} \cdot (6x-2)$ $\qquad$ put $u=3x^2-2x$ back

$= 4030(3x^2-2x)^{2014} \cdot (3x-1)$

Slogan : differentiate layer by layer.

Ex : Find the derivative of $\left(\frac{x}{x+1}\right)^2$.

(a) By chain rule ;

(b) Write $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$, then by quotient rule.

Ans : Both equal to $\frac{2x}{(x+1)^3}$.

e.g. Find the derivative of $e^{\sqrt{x^2+1}}$.

1st layer $y = e^w$ $w = \sqrt{x^2+1}$

2nd layer $w = \sqrt{u}$ $u = x^2+1$

3rd layer $u = x^2+1$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

e.g. Revisit of quotient rule.

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,[g(x)]^{-1}\right)$$

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\,\frac{d}{dx}[g(x)]^{-1} \qquad \text{(Product rule)}$$

Apply chain rule here

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\left\{-[g(x)]^{-2}\frac{dg}{dx}\right\}$$

$$= \frac{\frac{df}{dx}g(x) + f(x)\frac{dg}{dx}}{[g(x)]^2}$$

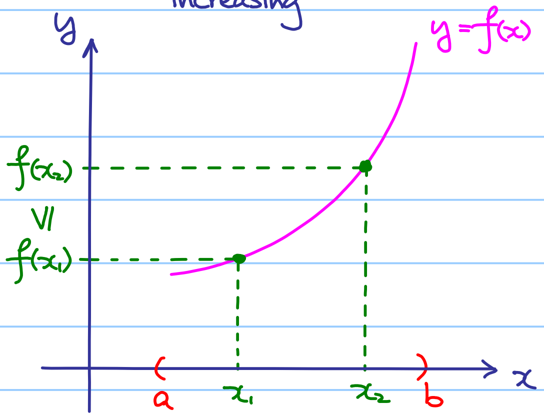
$$= \frac{f'(x)\,g(x) + f(x)\,g'(x)}{[g(x)]^2}$$

Increasing / Decreasing Functions

If $f(x)$ is a function such that for all x_1, x_2 with $a<x_1<x_2<b$, we have

† $f(x_1) \le f(x_2)$ $\quad$ ($f(x_1) \ge f(x_2)$) , then $f(x)$ is called an increasing (a decreasing) function on (a,b) .

Increasing



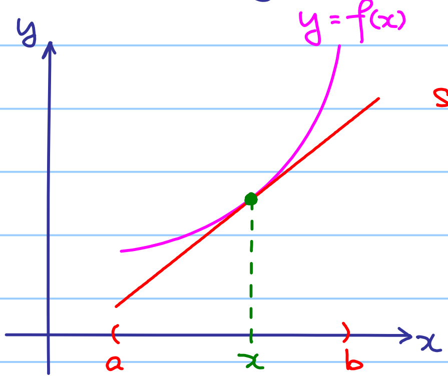
Roughly speaking :
The larger x we input
the larger y we get !

† If we have strict inequality , it is called a strictly increasing (decreasing) function on (a,b) .

FACT (without proof)

If $f(x)$ is differentiable on (a,b) and ${}^{\dagger\dagger}f'(x) \geq 0$ $(f'(x) \leq 0)$ for all $x \in (a,b)$, then $f(x)$ is increasing (decreasing) on (a,b).

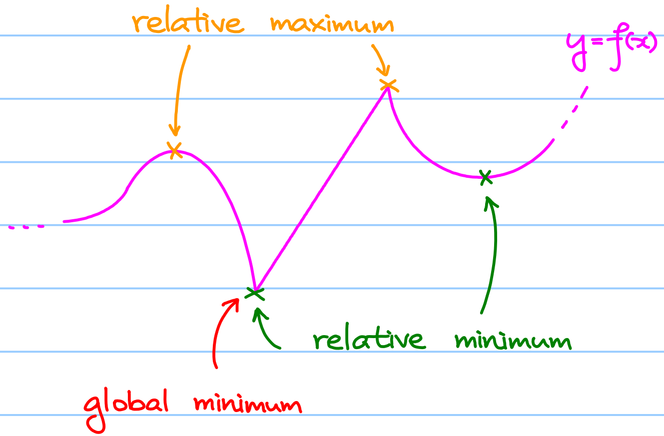
Increasing



†† If we have strict inequality, $f(x)$ is a strictly increasing (decreasing) function on (a,b).

Relative / Global Extrema :

Idea :



Note : No global maximum in this case .

f has a global maximum (resp. minimum) point at a if
$f(x) \leq f(a)$ (resp. $f(x) \geq f(a)$) for all x in the domain of f.

f has a relative maximum (resp. minimum) point at a if
$f(x) \leq f(a)$ (resp. $f(x) \geq f(a)$) for all x in a neighborhood of a.

Main question :
How differentiation helps to find relative / global extrema ?

e.g. Number of days of using drug : x

Life of a fish : T (weeks) which is estimated by

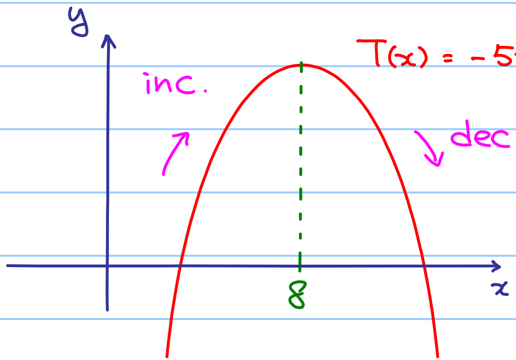
$T(x) = -5x^2 + 80x - 120$

$T'(x) = -10x + 80$

$T'(x) > 0$ | $T'(x) < 0$
$-10x + 80 > 0$ | $-10x + 80 < 0$
$x < 8$ | $x > 8$

$\therefore$ $T(x)$ is strictly increasing when $x < 8$ and
$T(x)$ is strictly decreasing when $x > 8$.

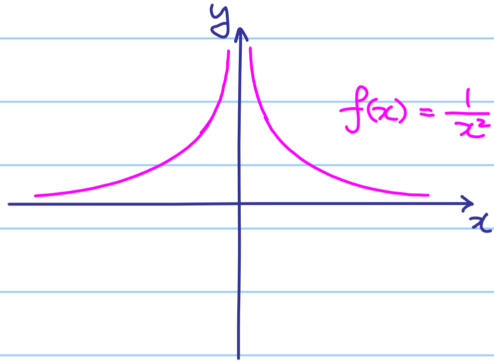
Not hard to understand why $T(x)$ attains maximum when $x = 8$
and maximum life of a fish $= T(8) = 200$ (weeks)



Note : $T'(8) = 0$.

Remark: Verify the above result by completing square.

e.g. Let $f(x) = \frac{1}{x^2}$, $x \neq 0$



$f'(x) = \frac{-2}{x^3}$

$f'(x) > 0$ if $x > 0$

$f'(x) < 0$ if $x < 0$

$\therefore$ $f(x)$ is strictly increasing when $x<0$

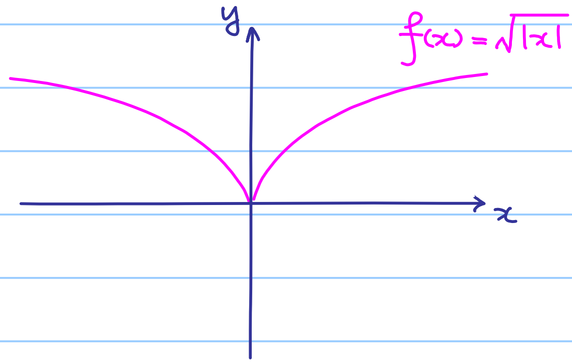
$f(x)$ is strictly decreasing when $x>0$

However, $f(0)$ is NOT well-defined, so there is NO maximum point.

e.g. Let $f(x) = \sqrt{|x|}$

Rewrite :

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$



If $x > 0$, $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}} > 0$

If $x < 0$, $f(x) = \sqrt{-x}$, then $f'(x) = -\frac{1}{2\sqrt{-x}} < 0$

$\therefore$ $f(x)$ is strictly increasing when $x > 0$

$f(x)$ is strictly decreasing when $x < 0$

However, $\lim\limits_{\Delta x \to 0^+} \dfrac{f(0+\Delta x)-f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \dfrac{\sqrt{\Delta x}}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \dfrac{1}{\sqrt{\Delta x}}$ which does NOT exist,

$\Rightarrow \lim\limits_{\Delta x \to 0} \dfrac{f(0+\Delta x)-f(0)}{\Delta x}$ does NOT exist

$\Rightarrow f'(0)$ does NOT exist

but as we can see f still attains minimum at $x=0$.

Exact statement :

1st Derivative Check :

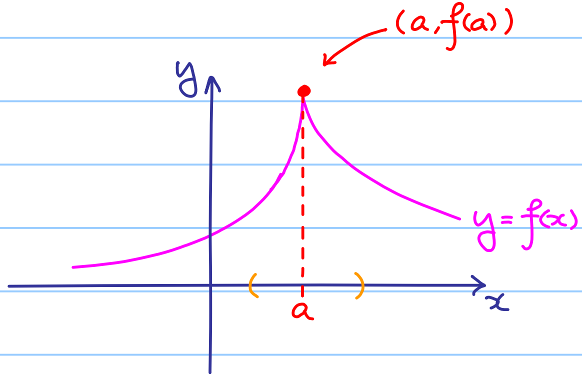
Suppose $f(x)$ is continuous at $x=a$ and differentiable on some neighborhood I containing a, except possibly at $x=a$ itself.

If $f'(x) \geq 0$ for all x in I with $x<a$, and

$f'(x) \leq 0$ for all x in I with $x>a$,

then $(a, f(a))$ is a relative maximum.

(Similar for relative minimum)



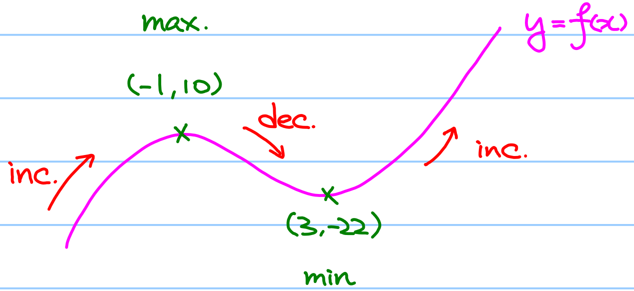
(Remember the slogan : Change sign of $f'(x)$ at $x=a$)

e.g. If $f(x) = x^3 - 3x^2 - 9x + 5$

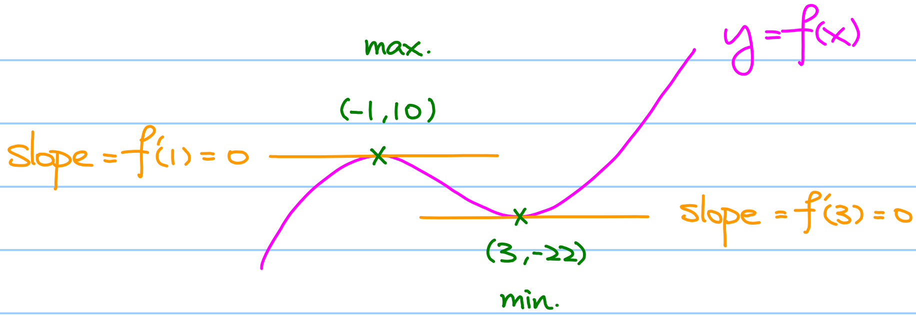
then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$



Furthermore,



Stationary Points :

If $f'(a) = 0$, then $(a, f(a))$ is called a stationary point .

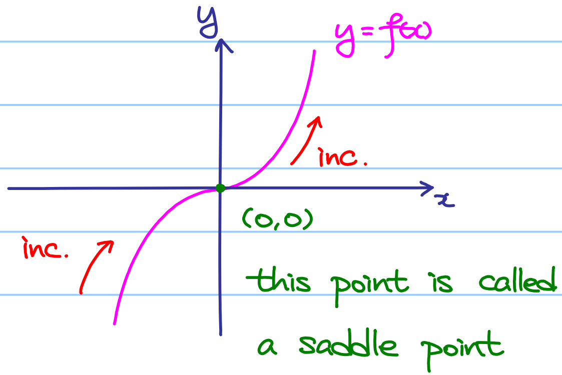
But even $f'(a) = 0$, it's still hard to say !

e.g. If $f(x) = x^3$, then $f'(x) = 3x^2$.

Note : 1) $f'(0) = 0$

2) $f'(x) = 3x^2 > 0$ for $x \neq 0$

i.e. No change of sign of $f'(x)$ at $x = 0$.



Note : a stationary is NOT necessary to be a max. / min. point !

Higher Derivatives :

$s(t)$: distance function (depends on time t)

(instantaneous) speed = rate of change of distance travelled with respect to t.

$$v(t) = \frac{ds}{dt}$$ (still a function of t)

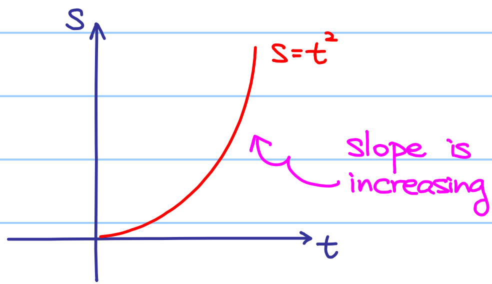
Question : What is $\frac{dv}{dt}$?

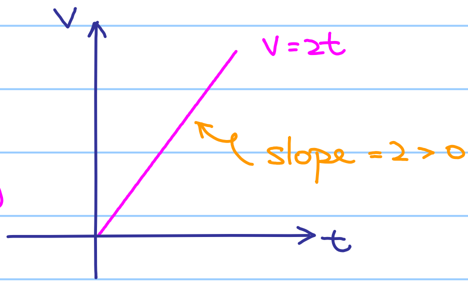
Answer : Acceleration !

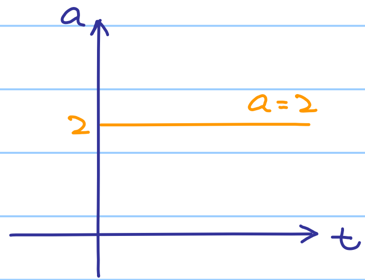
= rate of change of speed with respect to t.

We write $a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$

e.g. $s(t) = t^2$ $\qquad$ $v(t) = \frac{ds}{dt} = 2t$ $\qquad$ $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$







speed is increasing

i.e. accelerating

In general, let $y = f(x)$

We have : (1st derivative) $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

(2nd derivative) $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

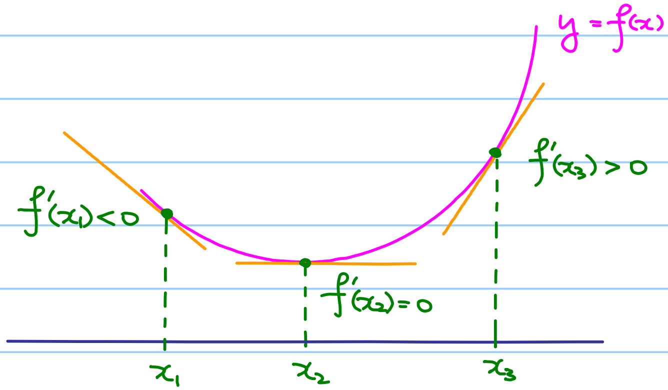
(nth derivative) $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

2nd Derivative and Concavity :

Think : If $f''(x) > 0$ for $a<x<b$

then $f'(x)$ is strictly increasing on (a,b)

Picture :



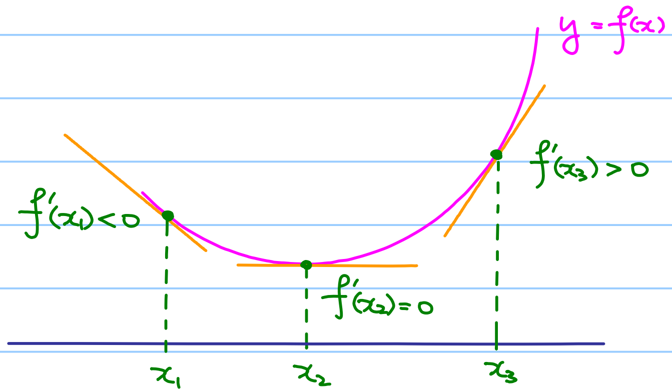
Slope of the tangent line at $(x,f(x))$ increases as x increases !

(NOT $f(x)$ is increasing !)

If $f''(x) > 0$ for $a<x<b$,

then $f(x)$ is a concave function on (a,b).

Picture :



If $f''(x) > 0$ for $a < x < b$,
then $f(x)$ is a concave function on (a, b).

Similarly : If $f''(x) < 0$ for $a < x < b$,
then $f(x)$ is a convex function on (a, b).

2nd Derivative Check :

Suppose $f(x)$ is twice differentiable at $x=a$. (i.e. $f'(a)$ and $f''(a)$ exist)

If (1) $f'(a)=0$ (i.e. $(a, f(a))$ is a stationary point.)

(2) $f''(a)<0$ (Roughly speaking : $f(x)$ is convex near $x=a$.)

then $(a, f(a))$ is a relative maximum.

We have similar result for relative minimum.

Caution : If $f''(a)=0$, then NO conclusion!

Consider $f(x)=x^4, x^3, -x^4$

We have $f'(0)=f''(0)=0$ in each case, but $(0,0)$ is

- min. for the 1st case.
- saddle point for the 2nd case.
- max. for the 3rd case.

e.g. If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

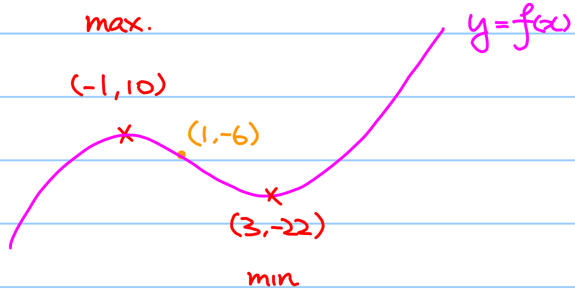
$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

$f''(x) = 6x - 6$

$f''(x) > 0$ if $x > 1$ $\qquad$ $f''(-1) = 12 < 0$

$f''(x) < 0$ if $x < 1$ $\qquad$ $f''(3) = 12 > 0$



$f'(x)$: +ve on the left of -1, −ve between -1 and 3, +ve on the right of 3

		-1		3	
$f'(x)$	+ve		−ve		+ve
$f(x)$	inc.		dec.		inc.

		1	
$f''(x)$	−ve		+ve
$f(x)$	convex		concave

Note: The curve changes from being convex to concave at (1, 6).

This point is called a point of inflection.

Point of inflection :

Suppose $f(x)$ is continuous at $x=a$ and differentiable on some open interval I containing $x=a$, except possibly at $x=a$ itself.

If $f''(x)>0$ (resp. $f''(x)<0$) for all x in I with $x<a$, and

$f''(x)<0$ (resp. $f''(x)>0$) for all x in I with $x<a$,

then $(a, f(a))$ is a point of inflection.

(Remember the slogan : Change sign of $f''(x)$ at $x=a$.)

e.g. $f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$

Find the range of x such that

(1) $f'(x) > 0$, $f'(x) < 0$

(2) $f''(x) > 0$, $f''(x) < 0$

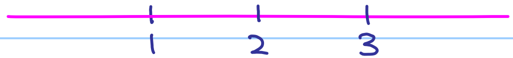
Step 1 : Find $f'(x)$ and factorize it.

$$f'(x) = 60x^4 - 420x^3 + 1020x^2 - 1020x + 360$$

$$= 60(x^4 - 70x^3 + 17x^2 - 17x + 6)$$

$$= 60(x-1)^2(x-2)(x-3)$$ (Using factor theorem)

Step 2:



$\downarrow$ gives intervals

$x<1 \qquad 1<x<2 \qquad 2<x<3 \qquad x>3$

(Reason: those factors may change sign at the boundaries of the intervals.)

Step 3 :

	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$(x-1)^2$	+	0	+	+	+	+	+
$(x-2)$	−	−	−	0	+	+	+
$(x-3)$	−	−	−	−	−	0	+
$f'(x)$	+	0	+	0	−	0	+
$f(x)$	inc	saddle pt.	inc.	max.	dec.	min	inc.

saddle point $= (1, -23)$

max $= (2, -16)$ min $= (3, -39)$

Similarly,

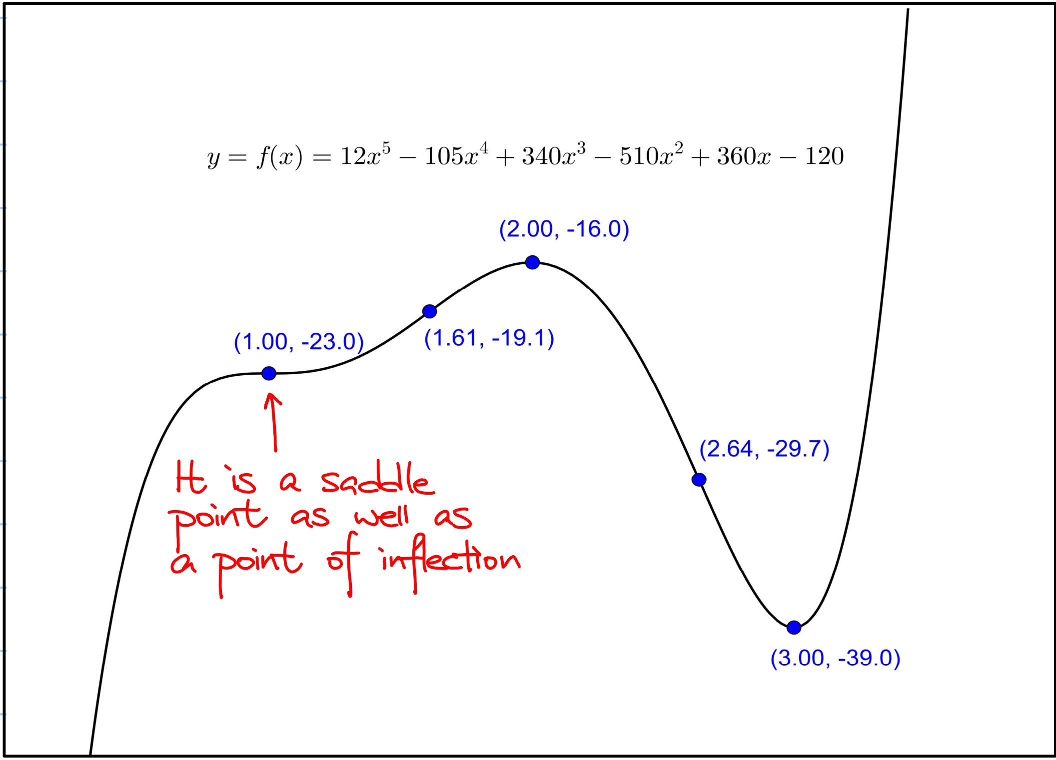
$$f''(x) = 240x^3 - 1260x^2 + 2040x - 1020$$

$$= 60(x-1)(4x^2 - 17x + 17)$$

$$= 240(x-1)\left[x - \left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x - \left(\frac{17-\sqrt{17}}{8}\right)\right]$$

$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
x		1		$\frac{17-\sqrt{17}}{8} \approx 1.61$		$\frac{17+\sqrt{17}}{8} \approx 2.64$	
$f(x)$	$\cap$		$\cup$		$\cap$		$\cup$

points of inflection : $(1, -23)$, $\left(\frac{17 \pm \sqrt{17}}{8}, f\left(\frac{17 \pm \sqrt{17}}{8}\right)\right)$



$y = f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$
(2.00, -16.0)
(1.00, -23.0)
(1.61, -19.1)
(2.64, -29.7)
(3.00, -39.0)
It is a saddle point as well as a point of inflection

e.g. $f(x) = \dfrac{x}{(x+1)^2}$ $\quad x \neq -1$

$f'(x) = \dfrac{1-x}{(x+1)^3}$

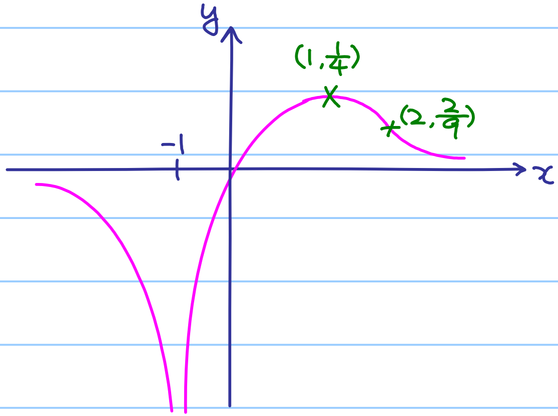
		-1		1	
$f'(x)$	$-$	NOT defined	$+$	0	$-$
$\Downarrow$					
$f(x)$	dec.	NOT defined	inc.	max.	dec.

max. $= (1, \frac{1}{4})$

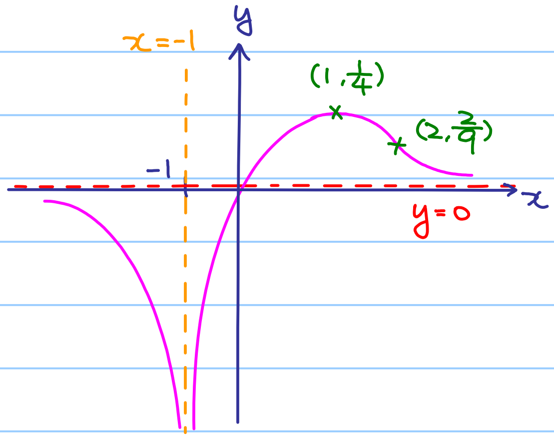
$$f''(x) = \frac{2(x-2)}{(x+1)^4}$$

		-1		2	
$f''(x)$	$-$	NOT defined	$-$	0	$+$
$\Downarrow$					
$f(x)$	$\cap$		$\cap$		$\cup$

point of inflection : $\left(2, \frac{2}{9}\right)$



y
x
-1
(1, 1/4)
(2, 2/9)



Note : The graph of $y=f(x)$ behaves like

- the vertical line $x=-1$, when x is "near" -1.
- the horizontal line $y=0$, when x is "near $+\infty$ or $-\infty$".

In fact, $x=-1$ is called a vertical asymptote,

$y=0$ is called a horizontal asymptote.

Finding vertical asymptote :

If $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^+} f(x) = +\infty$ or $-\infty$, then $x = a$ is called a vertical asymptote.

Finding horizontal asymptote :

If $\lim_{x \to +\infty} f(x) = L$, where L is a real number, then $y = L$ is a horizontal asymptote.
(Similar for $\lim_{x \to -\infty} f(x)$)

Note : It may happen that both $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ exist but they are NOT the same.

Curve Sketching :

Goal : Given a function $f(x)$, sketch the graph of $y = f(x)$.

(Capturing main features)

- x - intercept — solve $f(x) = 0$
- y - intercept — y-intercept $= f(0)$
- increasing / decreasing — solve $f'(x) > 0$ / $f'(x) < 0$

  saddle point / max. / min. — change of sign of $f'(x)$?
- concave / convex — solve $f''(x) > 0$ / $f''(x) < 0$

  point of inflection — change of sign of $f''(x)$?
- vertical asymptote — any $x = a$ with $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$
- horizontal asymptote — $\lim_{x \to +\infty} f(x)$, $\lim_{x \to -\infty} f(x)$ exist ?
- oblique asymptote (NOT covered)