Think : If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m > 0$ (i.e. $\deg p(x) = m$)

$q(x) = b_n x^n + a_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $b_n > 0$ (i.e. $\deg q(x) = n$)

then find $\lim\limits_{x \to +\infty} \dfrac{p(x)}{q(x)}$ for the following cases :

1) $\deg p(x) > \deg q(x)$ i.e. $m > n$

2) $\deg p(x) = \deg q(x)$ i.e. $m = n$

3) $\deg p(x) < \deg q(x)$ i.e. $m < n$

Ans :

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty & \text{if } \deg p(x) > \deg q(x) \\ \dfrac{a_m}{b_m} & \text{if } \deg p(x) = \deg q(x) \\ 0 & \text{if } \deg p(x) < \deg q(x) \end{cases}$$

Following this idea :

e.g. Find $\lim\limits_{x \to +\infty} \dfrac{3x-1}{\sqrt{4x^2+2}}$

$$\lim_{x \to +\infty} \frac{3x-1}{\sqrt{4x^2+2}}$$ ← roughly deg = 1

$$= \lim_{x \to +\infty} \frac{3-\frac{1}{x}}{\sqrt{4-\frac{2}{x}}}$$

$$= \frac{3}{2}$$

Constant e :

Consider a number $(1+\frac{1}{m})^n$ that depends on m and n and then

1) fix m, say $m=100$, n is getting larger and larger.

$n=10$	$n=100$	$n=1000$	$\rightarrow +\infty$
$(1+\frac{1}{m})^n = 1.01^{10}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.01^{1000}$	$\rightarrow +\infty$

2) fix n, say $n=100$, m is getting larger and larger.

$m=10$	$m=100$	$m=1000$	$\rightarrow +\infty$
$(1+\frac{1}{m})^n = 1.1^{100}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.001^{100}$	$\rightarrow 1$

How about setting $m=n$ and let them become larger and larger ?

$(1+\frac{1}{n})^n \to ?$ as $n \to +\infty$ (i.e. limit exists ?

something between $+\infty$ and 1 ??)

$n=10$	$n=100$	$n=1000$	$\to +\infty$
$(1+\frac{1}{n})^n = 1.1^{10}$	$(1+\frac{1}{n})^n = 1.01^{100}$	$(1+\frac{1}{n})^n = 1.001^{1000}$	
≈ 2.59374	≈ 2.70481	≈ 2.71692	$\to 2.71828\cdots$

FACT (without proof)

$\lim_{x\to+\infty}(1+\frac{1}{x})^x$ exists !

We define $e = \lim_{x\to+\infty}(1+\frac{1}{x})^x \approx 2.71828$ (i.e. call the limit e)

Limits Involving e :

e.g. Find $\lim_{x \to +\infty} \left(1+\frac{2}{x}\right)^x$

$$\lim_{x \to +\infty} \left(1+\frac{2}{x}\right)^x = \lim_{x \to +\infty} \left[\left(1+\frac{1}{\left(\frac{x}{2}\right)}\right)^{\frac{x}{2}}\right]^2$$

$$= e^2$$

(let $y = \frac{x}{2}$, as $x \to +\infty$, $y \to +\infty$

$\lim_{x \to +\infty} \left(1+\frac{1}{\left(\frac{x}{2}\right)}\right)^{\frac{x}{2}} = \lim_{y \to +\infty} \left(1+\frac{1}{y}\right)^y = e$)

e.g. Find $\lim_{x \to +\infty} \left(1 + \frac{1}{2x-1}\right)^x$

e.g. Find $\lim\limits_{x \to +\infty} \left(1+\frac{1}{2x-1}\right)^{x}$

$$\lim_{x \to +\infty} \left(1+\frac{1}{2x-1}\right)^{x} = \lim_{x \to +\infty} \left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$

$$= \lim_{x \to +\infty} \left[\left(1+\frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{1}{2}} \cdot \left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} \cdot 1$$

$$= e^{\frac{1}{2}}$$

e.g. Find $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x$

e.g. Find $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$

Let $y=-x$, as $x\to-\infty$, $y\to+\infty$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y}$$

$$= \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^{y}$$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)$$

$$= e\cdot 1$$

$$= e$$

Remark: From the above example, we know $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

e.g. Find $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$.

Let $y = \frac{1}{x}$, as $x \to 0$, $y \to \infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y} = e$$

Properties of e :

What so special with e ?

One is that : Consider a function $f(x) = e^x$

We have the expansion (Taylor Expansion) :

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad n! = 1 \times 2 \times 3 \times \cdots \times n$$

and we know $f'(x) = \frac{df}{dx} = f(x)$ (i.e. the derivative of e^x is itself)

(Discuss later !)

From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Roughly speaking : As $x \to +\infty$, e^x grows "faster" than any x^k, where $k>0$

FACT (without proof)

1) $\lim\limits_{x \to +\infty} x^k e^{-x} = \lim\limits_{x \to +\infty} \frac{x^k}{e^x} = 0$, for any $k>0$.

2) $\lim\limits_{x \to +\infty} p(x) e^{-x} = \lim\limits_{x \to +\infty} \frac{p(x)}{e^x} = 0$, for any polynomial $p(x)$.

Continunity :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{x \to c} f(x) = f(c)$.

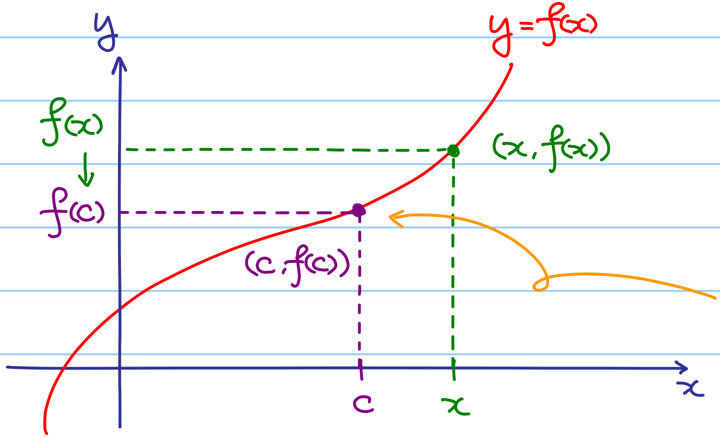
Idea :

③ they equal

$$\lim_{x \to c} f(x) = f(c)$$

① This limit exists

② f is well-defined at $x=c$

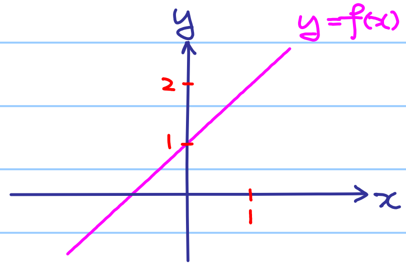


the curve does NOT break up at the point $x=c$!

If a function is continuous at every point,

then f is called a continuous function.

e.g. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x+1$



① $\lim_{x \to 1} f(x) = 2$

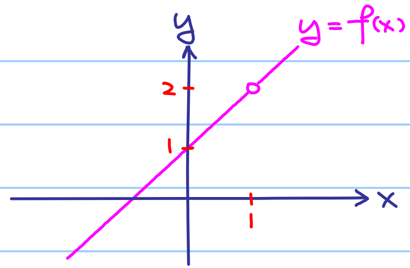
② $f(1) = 2$

$\therefore$ f is discontinuous at $x = 1$.

e.g. Let $f : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$.

① $\lim_{x \to 1} f(x) = 2$

② $f(1)$ is NOT well-defined.



$\therefore$ f is discontinuous at $x = 1$.

Recall :

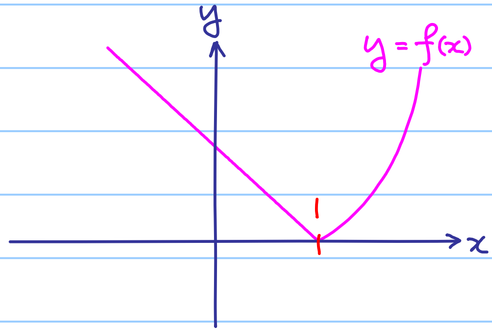
$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

Rewrite :

A function $f(x)$ is said to be continuous at $x = c$ if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

e.g. If $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$



① $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2 - 1 = 0$

② $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 1-x = 0$

③ $f(1) = 1^2 - 1 = 0$

$\therefore$ f is continuous at $x = 1$.

Absolute value :

$$|x| \stackrel{\text{def}}{=} \sqrt{x^2}$$

e.g. $|3| = \sqrt{3^2} = \sqrt{9} = 3$

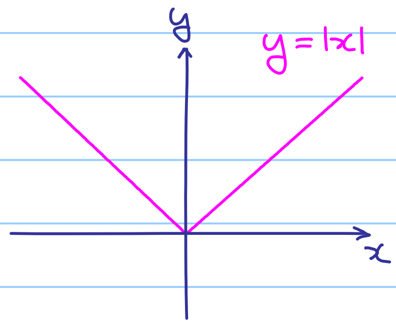
$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

$|0| = 0$

(Simply speaking : throw away the + or − sign)

Rewrite :

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



e.g. Prove $f(x) = |x|$ is continuous at $x = 0$.

① $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

② $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0$

③ $f(0) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Further question: Is $f(x) = |x|$ a continuous function?

Remarks :

1) We can further rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{h\to 0} f(c+h) = f(c)$

(Hint : let $x=c+h$, as $h\to 0$, $x\to c$)

2) FACT (Without proof)

- polynomial function $p(x)$ is continuous everywhere .
- $\sqrt{x}$ is continuous for $x \geq 0$
- If $f(x)$, $g(x)$ are continuous , then $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ (when $g(x) \neq 0$) are continuous.
- If $f(x)$, $g(x)$ are continuous , then $f(g(x))$ (when it is defined) is continuous .

e.g. $f(x) = \dfrac{2x^2+3}{x^2-3x+2}$ quotient of two polynomials (continuous functions)

$= \dfrac{2x^2+3}{(x-2)(x-1)}$ the denominator is nonzero when $x \neq 1$ or 2

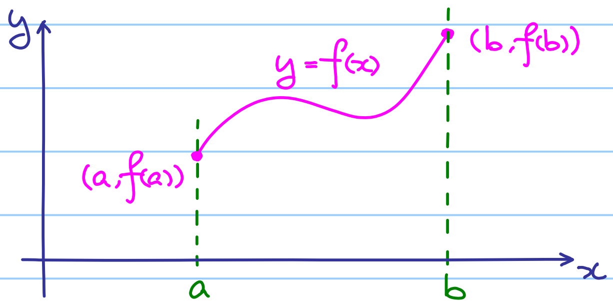
$\therefore$ $f(x)$ is continuous everywhere except $x = 1, 2$

Continuous on $[a,b]$:

Let $f:[a,b]\to\mathbb{R}$ be a function,

f is said to be continuous at $x=a$ if $\lim_{x\to a^+} f(x) = f(a)$

f is said to be continuous at $x=b$ if $\lim_{x\to b^-} f(x) = f(b)$

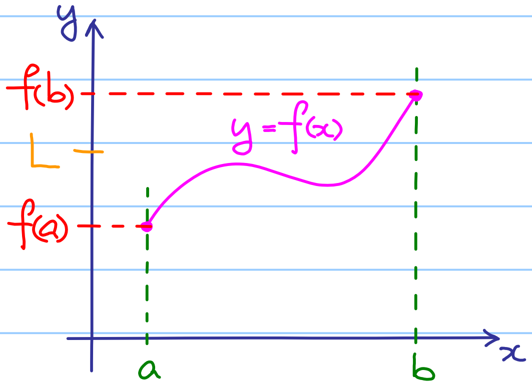


(We cannot talk about $\lim_{x\to a^-} f(x)$ and $\lim_{x\to b^+} f(x)$!)

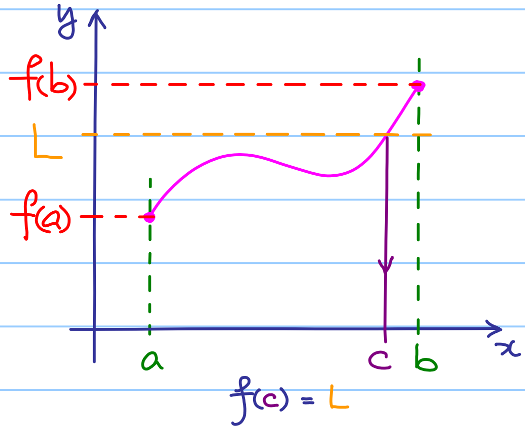
If $f:[a,b]\to\mathbb{R}$ is continuous at every point $x\in[a,b]$,
then f is said to be continuous on $[a,b]$.

Mean Value Property :

Suppose that f is continuous on $[a,b]$ and $f(a) < f(b)$.
Furthermore, if L is a real number such that $f(a) < L < f(b)$,
then there exists (at least one) $c \in (a,b)$ such that $f(c) = L$.



$\Longrightarrow$



Similar result holds for $f(a) > L > f(b)$. (What is the picture ?)

e.g. x : Number of products produced (in hundreds units)

Revenue $= R(x) = 100x(400 - 3x^2)$

Cost $= C(x) = 120000 + 700x$

Profit $= P(x) = R(x) - C(x) = 100(-3x^3 + 393x - 1200)$

① $P(0) = -120000 < 0$

② $P(5) = 39000 > 0$

③ $P(x)$ is a polynomial, so it is continuous everywhere,
In particular, it is continuous on $[0, 5]$

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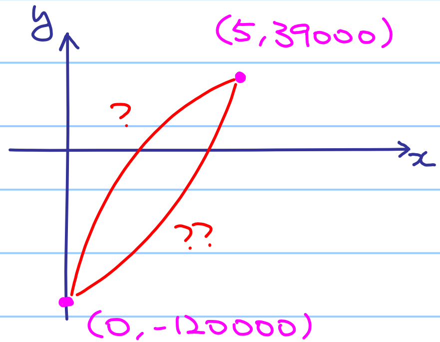


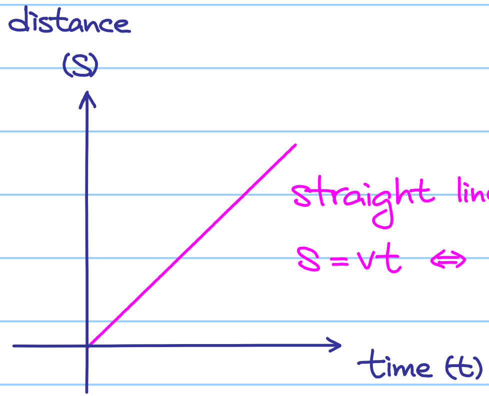
We do NOT know the shape of the graph, but we know it intersects the x-axis at least once.

i.e. $P(c) = 0$ (which means breakeven)

for some $c \in (0, 5)$

Differentiation :

Recall : (average) speed $= \dfrac{\text{distance}}{\text{time}}$



straight line

$s = vt \iff v = \dfrac{s}{t}$

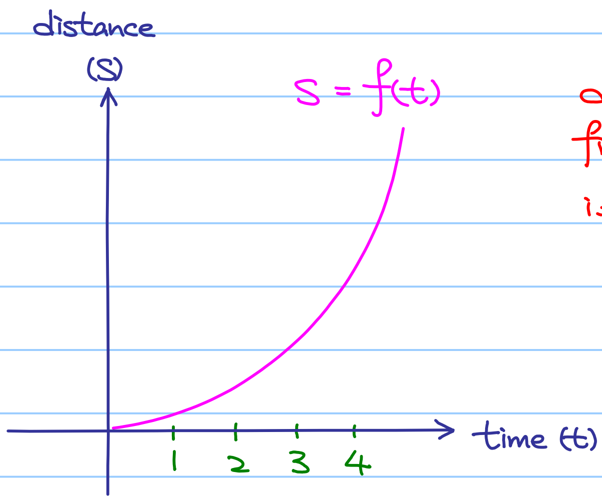
Note : constant speed !

speed = slope = v

Remark :

Using displacement and velocity if you know .

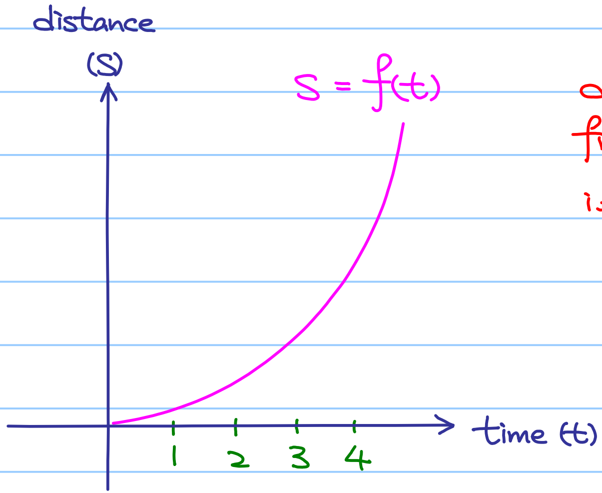
How about this case ?



distance traveled from t=0 to t=1 < distance traveled from t=3 to t=4

i.e. speed is changing

Speed is different at different moment.



distance traveled from $t=0$ to $t=1$ < distance traveled from $t=3$ to $t=4$

i.e. speed is changing

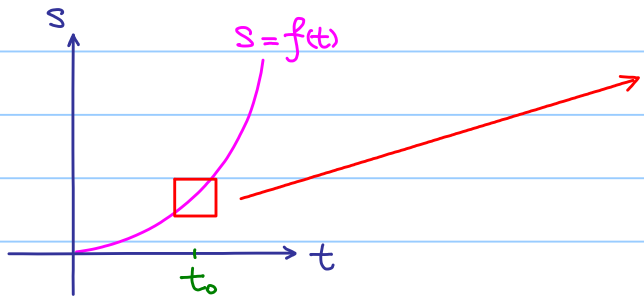
Speed is different at different moment.

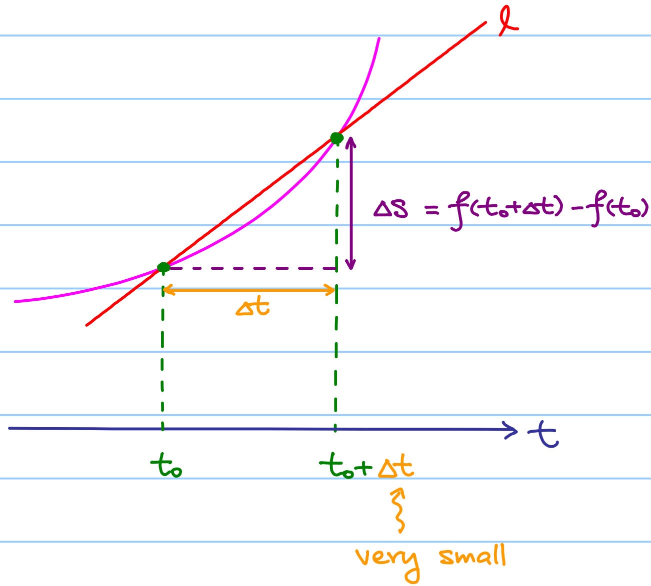
Hold on!

What is the meaning of speed at a particular moment (instantaneous speed)?

We need a definition!

Instantaneous speed at $t = t_0$:





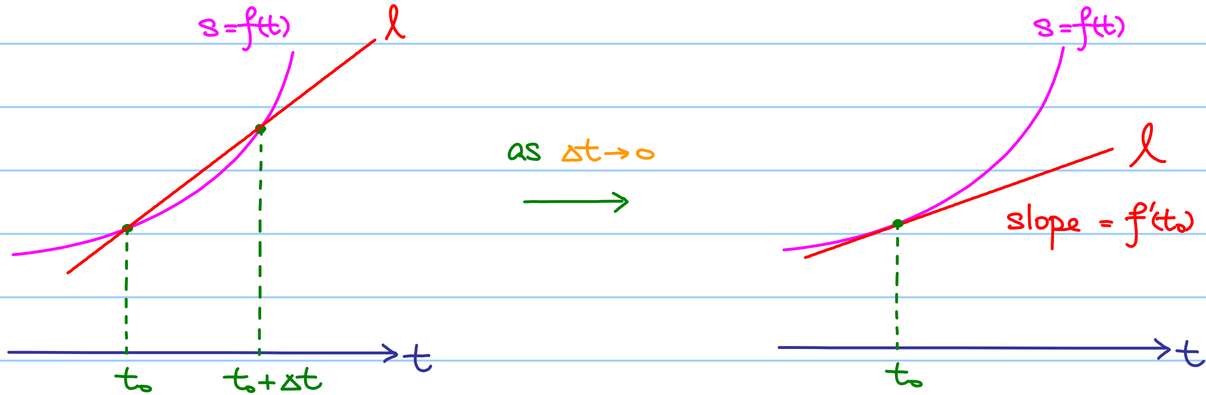
Average speed between t_0 and $t_0 + \Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \text{slope of } \ell$$

Idea : Let Δt becomes smaller and smaller !

Instantaneous speed at $t=t_0$ is defined to be $\lim\limits_{\Delta t \to 0} \dfrac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$

(provided it exists , if so , it is denoted by $f'(t_0)$)



Note : When $\Delta t \to 0$, ℓ becomes the tangent line at $t=t_0$, so

slope of the tangent line at $t=t_0$ $= f'(t_0)$

e.g. If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4\Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 4 + \Delta t = 4$$

In general, we have $y = f(x)$, fix x_0.

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

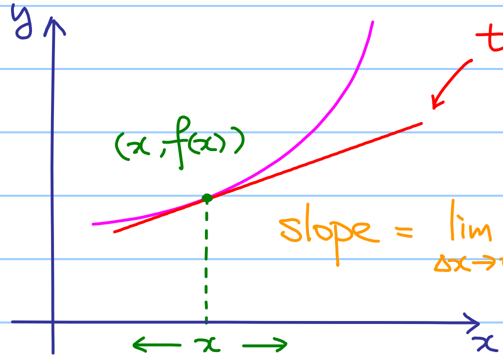
$f(x)$ is said to be differentiable at $x = x_0$ if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ exists} \quad (\text{denoted by } f'(x_0))$$

It is called the derivative of $f(x)$ at $x = x_0$.

Note: By definition, if $f(x_0)$ is NOT well-defined, we cannot define $f'(x_0)$, so $f(x)$ must NOT be differentiable at $x = x_0$.

Perform the pervious step to different points :



tangent line at $(x, f(x))$

$$\text{slope} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Recall : What is a function ?

Roughly speaking , given an input x , return a value .

Now, we construct a new function , $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

(i.e. given an input x, return the slope of the tangent line at $(x, f(x))$)

e.g. If $f(x) = x^2$, find $f'(x)$

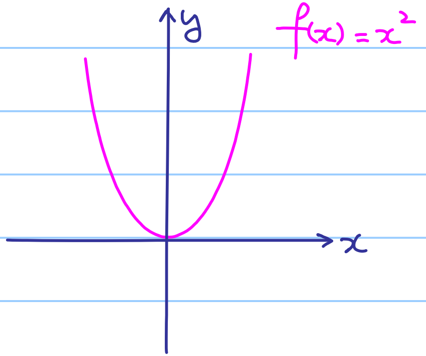
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

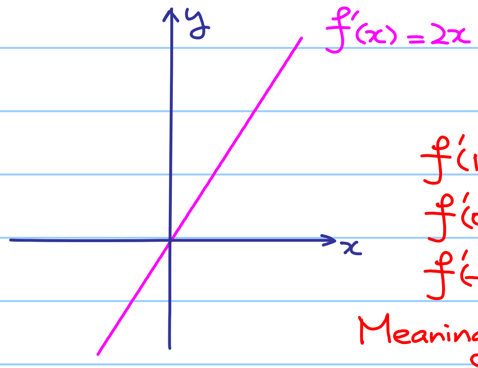
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:





$f'(1) = 2$

$f'(0) = 0$

$f'(-1) = -2$

Meaning ??

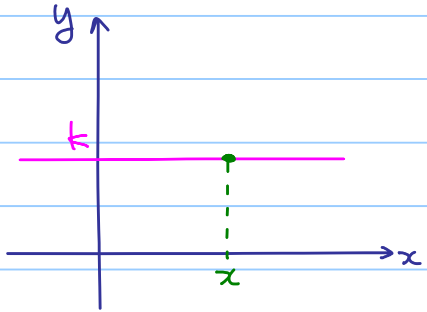
Notations :

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

$$\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} = f'(3) = 2(3) = 6$$

e.g. If $f(x) = k$, where k is a constant , $f'(x) = ?$



Note : Slope of the tangent line at $(x, f(x)) = (x, k)$ is zero.

$\therefore f'(x) = 0$

Concrete computation :

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k - k}{\Delta X} \quad (\Delta x \neq 0)$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 \quad = 0$$

Ex: Find $f'(x)$ if

(a) $f(x) = x$ Ans: $f'(x) = 1$

(b) $f(x) = x^3$ $f'(x) = 3x^2$

FACT (Without proof)

If $f(x) = x^r$, where r is a real number,

then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think: If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$)

FACT: If $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.

proof: By assumption, $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ exists.

Also, we know $\lim_{\Delta x \to 0} \Delta x = 0$

$$\lim_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = \lim_{\Delta x \to 0} \left(\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right)$$

both exist

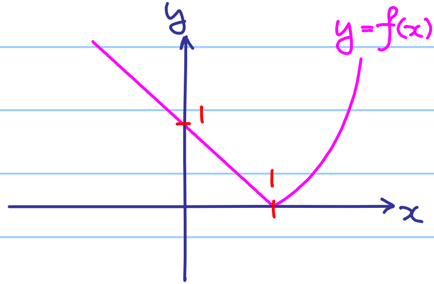
$$= \left(\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \right) \cdot \left(\lim_{\Delta x \to 0} \Delta x \right)$$

$$= f'(x_0) \cdot 0 = 0$$

$\therefore \lim_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0)$, so $f(x)$ is continuous at $x = x_0$.

However, the converse is NOT true.

e.g. If $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$



$$\lim_{\Delta x\to 0^+}\frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x\to 0^+}\frac{[(1+\Delta x)^2-1]-[1^2-1]}{\Delta x} = \lim_{\Delta x\to 0^+}\frac{2\Delta x+\Delta x^2}{\Delta x} = 2$$

(it means we are looking at small but positive Δx)

$$\lim_{\Delta x\to 0^-}\frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{[1-(1+\Delta x)]-[1^2-1]}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{-\Delta x}{\Delta x} = -1$$

(it means we are looking at small but negative Δx)

$$\lim_{\Delta x\to 0^+}\frac{f(1+\Delta x)-f(1)}{\Delta x} \neq \lim_{\Delta x\to 0^-}\frac{f(1+\Delta x)-f(1)}{\Delta x} \Rightarrow \lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x} \text{ does NOT exist !}$$

$\therefore f(x)$ is NOT differentiable at $x=1$.

Ex:

a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim\limits_{x\to 1} f(x) = f(1)$.

(Therefore, $f(x)$ is continuous at $x=1$, but NOT differentiable at $x=1$)

b) Show that $f(x)$ is differentiable everywhere except $x=1$, and

$$f'(x) = \begin{cases} 2x & \text{if} \quad x>1 \\ \text{undefined} & \text{if} \quad x=1 \\ -1 & \text{if} \quad x<1 \end{cases}$$

Differentiable function: If $f(x)$ is differentiable everywhere, then $f(x)$ is said to be a differentiable function.