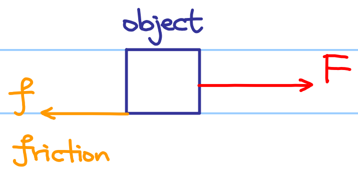
Work done (energy)

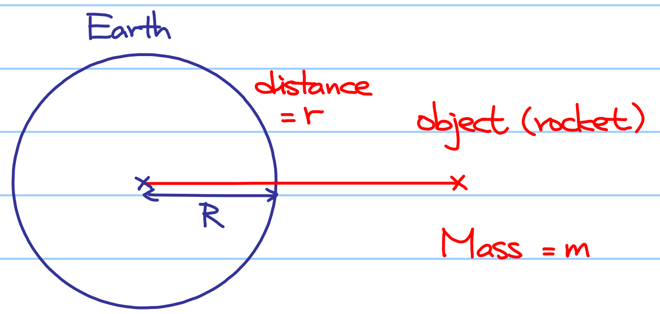


moves with constant velocity $\Rightarrow$ $F = f$

distance traveled $= s$

Work done by F against friction $= F \cdot s$

(energy)



Mass $= m$

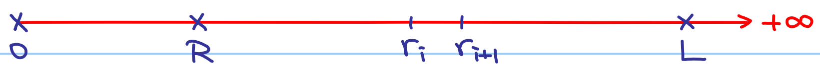
Mass $= M$

radius $= R$

Gravitational Force $= \dfrac{GMm}{r^2}$

Remark :

- $G = 6.67 \times 10^{-11}\ (m^{-3} kg^{-1} s^{-2})$



$\Delta r = r_{i+1} - r_i$

Energy to bring the rocket from r_i to r_{i+1}

$\approx \dfrac{GMm}{r_i^2} \Delta r$

Energy to bring the rocket from R to L

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{GMm}{r_i^2} \Delta r$$

$$= \int_R^L \frac{GMm}{r^2} dr$$

$$= \left[\frac{-GMm}{r}\right]_R^L$$

$$= GMm\left(-\frac{1}{L} + \frac{1}{R}\right)$$

Energy to bring the rocket from R to $+\infty$ (Escape from the Earth)

$$= \int_R^{+\infty} \frac{GMm}{r^2} dr$$

$$= \lim_{L\to+\infty} \int_R^L \frac{GMm}{r^2} dr$$

$$= \lim_{L\to+\infty} \left[\frac{-GMm}{r}\right]_R^L$$

$$= \lim_{L\to+\infty} GMm\left(-\frac{1}{L} + \frac{1}{R}\right)$$

$$= \frac{GMm}{R}$$

Recall: Conservation of energy

$\Rightarrow$ initial kinetic energy = energy for the rocket to escape

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

$$v = \sqrt{\frac{2GM}{R}}$$ called escape velocity

i.e. minimum velocity to launch the rocket

Differential Equation

A differential equation is an equation that involves some function of one or more variables with its derivatives.

e.g. $\frac{dy}{dx} = 3x^2+5$, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 7x+3$

$3\frac{\partial f(x,y)}{\partial x} + 2\frac{\partial f(x,y)}{\partial y} = 3xy$, ...

What we do here :

First order ordinary differential equation (1st order ODE)

Ordinary : Single variable , i.e. y depends on x only.

1st order : Involving derivatives up to 1st order , i.e. $\frac{dy}{dx}$ at most

i.e. $F(x,y,\frac{dy}{dx}) = 0$.

Solving the ODE $F(x,y,\frac{dy}{dx}) = 0$ means finding $y(x)$.

- $\frac{dy}{dx} = g(x)$

Simplest one : $y(x) = \int g(x)\, dx$

e.g. solve $\frac{dy}{dx} = 3x^2+5$

$$y = \int 3x^2+5\, dx$$

$$y = x^3 + 5x + C$$

- separable equation $\frac{dy}{dx} = \frac{h(x)}{g(y)}$

method : $g(y)dy = h(x)dx$

$$\int g(y)dy = \int h(x)dx$$

e.g. $\frac{dy}{dx} = \frac{2x}{y^2}$

$$y^2 dy = 2x dx$$

$$\int y^2 dy = \int 2x \, dx$$

$\frac{1}{3}y^3 + C_1 = x^2 + C_2$ let $C' = C_2 - C_1$

(This step can be skipped.)

$$\frac{1}{3}y^3 = x^2 + C'$$

$y^3 = 3x^2 + C$ (let $C = 3C'$)

$$y = (3x^2 + C)^{\frac{1}{3}}$$

e.g. Spread of rumor.

Population of a school = 200

t : time (day)

Assumption : each one who knows the rumor would talk to 5 people each day

$x(t)$: number of people who know the rumor at time t.

$x(0) = 20$

$\frac{dx}{dt}$ = rate of increase of people who know the rumor.

= 5 × # people who know the rumor at time t

× probability of meeting a person who does NOT know the rumor

$= 5x\left(\frac{200-x}{200}\right)$

$= \frac{x(200-x)}{40}$

$$\frac{dx}{dt} = \frac{x(200-x)}{40}$$

$$\int \frac{40}{x(200-x)}\,dx = \int dt$$

$$\frac{1}{5}\ln\left|\frac{x}{200-x}\right| = t + C$$

put $t=0$, $x=20$: $C = \frac{1}{5}\ln\frac{1}{9}$

$$\frac{1}{5}\ln\left|\frac{x}{200-x}\right| = t + \frac{1}{5}\ln\frac{1}{9}$$

$$\frac{200}{x} - 1 = 9e^{-5t}$$

$$x = \frac{200}{1+9e^{-5t}}$$

Remark :

1) $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \frac{200}{1+9e^{-5t}} = 200$. Eventually, everybody knows.

2) How long does it take so that half of people know the rumor ?

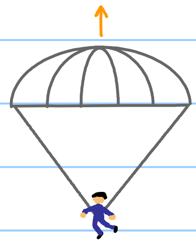
e.g. Parachute

Recall : Newton 2nd Law of motion

$$F = ma = m\frac{dv}{dt}$$

Net force   mass   acceleration

air resistance = $-kv$



$\downarrow$ +ve

gravitational force $\downarrow mg$ $g = 9.8\,ms^{-2}$

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{m}{mg-kv}\,dv = dt$$

$$\int \frac{m}{mg-kv}\,dv = \int dt$$

$$-\frac{m}{k}\ln|mg-kv| = t + C_1$$

When $t=0$, $v=v_0$

v_0 is called initial velocity.

$$|mg-kv| = C_2 e^{-\frac{k}{m}t} \quad (C_2 = e^{-\frac{kC_1}{m}} > 0)$$

$$mg - kv = \pm C_2 e^{-\frac{k}{m}t}$$

$$v = \frac{mg}{k} \mp \frac{C_2}{k} e^{-\frac{k}{m}t}$$

$$v(0) = v_0 \Rightarrow v_0 - \frac{mg}{k} = \mp \frac{C_2}{k}$$

(Which sign should we pick ?)

Case 1 : $v_0 > \frac{mg}{k}$

$0 < v_0 - \frac{mg}{k} = \mp \frac{C_2}{k}$

$\frac{C_2}{k} > 0 \Rightarrow$ we should pick the green one.

$\therefore C_2 = +(kv_0 - mg)$ and $v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k})e^{-\frac{k}{m}t}$

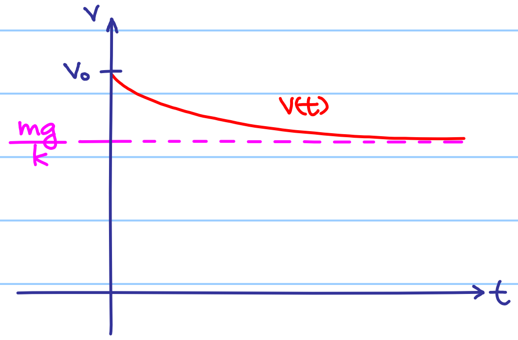
Case 2 : $v_0 < \frac{mg}{k}$

$0 > v_0 - \frac{mg}{k} = \mp \frac{C_2}{k}$

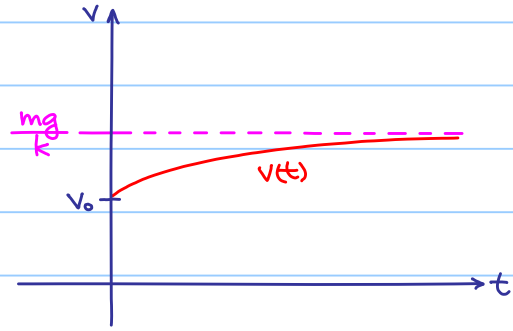
$\frac{C_2}{k} > 0 \Rightarrow$ we should pick the red one.

$\therefore C_2 = -(kv_0 - mg)$ and $v(t) = \frac{mg}{k} - (\frac{mg}{k} - v_0)e^{-\frac{k}{m}t}$

Case 1 : $v_0 > \frac{mg}{k}$



Case 2 : $v_0 < \frac{mg}{k}$



Remark :

1) $\lim\limits_{t \to +\infty} v(t) = \frac{mg}{k}$

$\therefore$ Terminal velocity $= \frac{mg}{k}$

2) Terminal velocity is independent from the initial velocity v_0.

First Order Linear Ordinary Differential Equations :

All 1st order linear ODEs are of the form :

$$\frac{dy}{dx} + p(x)y = q(x)$$

Regard $\left(\frac{d}{dx} + p(x)\right)$ as an operator (differential operator) acting on $y(x)$.

i.e. $\left(\frac{d}{dx} + p(x)\right) y(x) = \frac{dy}{dx} + p(x)y$

"Linear" means the following properties :

i) $\left(\frac{d}{dx} + p(x)\right)(y_1(x) + y_2(x)) = \left(\frac{d}{dx} + p(x)\right) y_1(x) + \left(\frac{d}{dx} + p(x)\right) y_2(x)$

ii) $\left(\frac{d}{dx} + p(x)\right)(c \cdot y(x)) = c \cdot \left(\frac{d}{dx} + p(x)\right) y(x)$, where c is a constant.

(Some advantages in ODE theory if one is linear.)

If $q(x) = 0$, then it is called a homogeneous equation,
otherwise it is called imhomogeneous equation.
(Usually, homogeneous equations are easier to be solved.)

Note that if $q(x) = 0$, then

$$\frac{dy}{dx} + p(x)y = 0$$

$$\frac{dy}{dx} = -p(x)y \quad \text{(separable equation)}$$

$$\frac{1}{y}dy = -p(x)dx$$

$$\int \frac{1}{y}dy = -\int p(x)dx$$

$$\ln|y| = -\int p(x)dx$$

$$y = \pm \exp\left(-\int p(x)dx\right)$$

Note that if $q(x) \neq 0$, then it is no longer separable!

How to solve $\frac{dy}{dx} + p(x)y = q(x)$?

Idea: Can we express $\frac{dy}{dx} + p(x)y$ into $\frac{d}{dx}(?)$?

If yes, then
$$\frac{dy}{dx} + p(x)y = q(x)$$
$$\frac{d}{dx}(?) = q(x)$$
$$? = \int q(x)\,dx$$

Unfortunately, we cannot, but how about multiplying a function $I(x)$ on both sides and try again?

$$\underbrace{I(x)\frac{dy}{dx} + I(x)p(x)y}_{\text{like result obtained by product rule}} = I(x)q(x)$$

$$\frac{d}{dx}(I \cdot y) = I\frac{dy}{dx} + \frac{dI}{dx}y$$

$\therefore$ The only question left is how to get a function $I(x)$ so that
$$\frac{dI}{dx}y = I(x)p(x)y$$
$$\frac{dI}{dx} = I(x)p(x)$$

$$\int \frac{1}{I}\,dI = \int p(x)\,dx$$
$$\ln I = \int p(x)\,dx$$

Ans: Let $I(x) = e^{\int p(x)dx}$, it is perfect!

$I(x)$ is called an integrating factor.

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}q(x)$$

$$\frac{d}{dx}\left(e^{\int p(x)dx}y\right) = e^{\int p(x)dx}q(x)$$

$$\frac{d}{dx}(I(x)y) = I(x)q(x)$$ (Note : $I(x) = e^{\int p(x)dx}$)

$$I(x)y = \int I(x)q(x)\,dx$$

$$y = \frac{1}{I(x)}\int I(x)q(x)\,dx$$

e.g. Solve $\frac{dy}{dx} + \frac{y}{x} = e^{-x}$

Note : $p(x) = \frac{1}{x}$ and $q(x) = e^{-x}$

$$I(x) = e^{\int \frac{1}{x}dx}$$
$$= e^{\ln x}$$
$$= x$$

Remark :

$\int \frac{1}{x}dx = \ln x + C$. Why we choose $C = 0$?

Check : $C \neq 0$, just multiply both sides by a constant.

Multiply both sides by $I(x) = x$:

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

$$x\frac{dy}{dx} + y = xe^{-x}$$

$$\frac{d}{dx}(xy) = xe^{-x}$$

$$xy = \int xe^{-x}dx$$

Ex : Integration by parts

$$= -e^{-x}(x+1) + C$$

DO NOT forget !

$$y = \frac{1}{x}[-e^{-x}(x+1) + C]$$

e.g. $\frac{dy}{dx} = 1 + x + y + xy$

$$\frac{dy}{dx} - (1+x)y = 1+x$$

Note: $p(x) = -(1+x)$

$I(x) = e^{\int p(x)dx} = e^{-\int 1+x dx} = e^{-(x+\frac{x^2}{2})}$

$$\frac{dy}{dx} - (1+x)y = 1+x$$

$$e^{-(x+\frac{x^2}{2})}\frac{dy}{dx} - e^{-(x+\frac{x^2}{2})}(1+x)y = e^{-(x+\frac{x^2}{2})}(1+x)$$

$$\frac{d}{dx}\left(e^{-(x+\frac{x^2}{2})}y\right) = e^{-(x+\frac{x^2}{2})}(1+x)$$

$$e^{-(x+\frac{x^2}{2})}y = \int e^{-(x+\frac{x^2}{2})}(1+x)\,dx$$

$$e^{-(x+\frac{x^2}{2})}y = -e^{-(x+\frac{x^2}{2})} + C$$

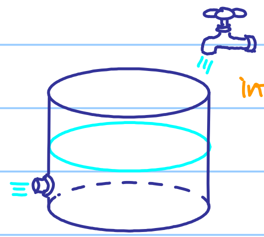
$$y = Ce^{(x+\frac{x^2}{2})} - 1$$

Ex: By writing $\frac{dy}{dx} = 1 + x + y + xy = (1+x)(1+y)$ as a separable equation, try to solve it and compare the results.

Application (setting up ODE)

e.g. (P720, Ex 9.2, Q53)

(Dilution) A tank contains 5 pounds of salt dissolved in 40 gallons of water. Pure water runs into the tank at the rate of 1 gal/min, and the mixture, kept uniform by stirring, runs out at the rate of 3 gal/min.



Set up and solve an initial value problem for the amount of salt $S(t)$ in the tank at time t.

Key point : Everything is clear except this :

What is the rate of change of salt ? (Of course, decreasing!)

What happens at time t:

Amount of water $= 40-(3-1)t = 40-2t$ gal

$(\therefore\ 0 \le t \le 20)$

Amount of salt $= S(t)$ pound

$\therefore$ Concentration $= \dfrac{S(t)}{40-2t}$ pound / gal

Rate of change of salt $= -$ rate of outflow water $\times$ concentration

minus sign indicates decreasing

$$\frac{dS}{dt} = -\ 3\ \cancel{\text{gal}}/\text{min} \times \frac{S(t)}{40-2t}\ \text{pound}/\cancel{\text{gal}}$$

$$\frac{dS}{dt} = -\frac{3S}{40-2t} \qquad \text{(Simply drop } t \text{ in writing } S\text{)}$$

(separable equation)

If we rewrite the equation as $\dfrac{dS}{dt} + \left(\dfrac{3}{40-2t}\right)S = 0$,

it is in fact a homogeneous 1st order linear ODE

$$\frac{dS}{dt} = -\frac{3S}{40-2t}$$

$$\frac{1}{S}dS = -\frac{3}{40-2t}dt$$

$$\int \frac{1}{S}dS = -3\int \frac{1}{40-2t}dt$$

$$\ln S = \frac{3}{2}\ln(40-2t)+C'$$

Think : Why S, $40-2t \geq 0$?

$$S(t) = C\,(40-2t)^{\frac{3}{2}}$$

Initial condition : $S(0) = 5$

$$\Rightarrow \quad 5 = C\,40^{\frac{3}{2}} \quad \Rightarrow \quad C = 5\cdot 40^{-\frac{3}{2}}$$

$\therefore \; S(t) = \dfrac{1}{8\sqrt{5}}(20-t)^{\frac{3}{2}}$ for $0 \leq t \leq 20$

Let's look at this :

$$\frac{1}{S}dS = -\frac{3}{40-2t}dt$$

$$\int \frac{1}{S}dS = \int \frac{3}{2t-40}dt$$

$\ln S = \frac{3}{2}\ln(2t-40)+C'$ $\xleftarrow{\text{Wrong}}$ $\because\; 2t-40<0$

$S = C\,(2t-40)^{\frac{3}{2}}$

we should write :

$\ln S = \frac{3}{2}\ln|2t-40| + C'$

then $\ln S = \frac{3}{2}\ln(40-2t) + C'$

as $|2t-40| = 40-2t$

When we put $t=0$,

we have $3(-40)^{\frac{3}{2}}$ which is NOT defined !

What's wrong? How to correct ?

e.g. (93 HKAL Applied Math)

A patient takes 100 mg of an orally administered drug, which will be gradually absorbed by the body and then eventually excreted out of the body. After t hours, let

x mg be the amount of drug still unabsorbed,

y mg be the amount of drug absorbed and still remaining in the body,

z mg be the amount of drug excreted out of the body.

The total amount of drug, $x+y+z$ remains constant over time. It is known that x decreases at a rate $\frac{2}{5}x$ mg per hour and

z increases at a rate $\frac{2}{25}y$ mg per hour.

At $t=0$, $x=100$ and $y=z=0$.

a) Show that $x=100e^{-\frac{2}{5}t}$.

b) Show that $\frac{dy}{dt}+\frac{2}{25}y=40e^{-\frac{2}{5}t}$

Hence find y and z in terms of t.

c) Find the time at which the amount of drug absorbed and still remaining in the body is at a maximum.

a) $\frac{dx}{dt}=-\frac{2}{5}x$

$$\int \frac{1}{x}dx=\int -\frac{2}{5}dt$$

$$\ln x = -\frac{2}{5}t+C'$$

$$x = Ce^{-\frac{2}{5}t}$$

Initial condition $x(0)=100 \Rightarrow C=100$

$\therefore x=100e^{-\frac{2}{5}t}$

b) (The most difficult part)

Want to obtain an equation involving $\frac{dy}{dt}$, but no direct information

Method: Relate $\frac{dy}{dt}$ to $\frac{dx}{dt}$ and $\frac{dz}{dt}$ which are known!

How?

$$x+y+z=100$$

$$\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}=0$$

$$\frac{dx}{dt} = -\frac{2}{5}x = -\frac{2}{5}\cdot 100e^{-\frac{2}{5}t} = -40e^{-\frac{2}{5}t}$$ OK!

$$\frac{dz}{dt} = \frac{2}{25}y$$ OK!

$$\therefore \quad \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$$

$$-40e^{-\frac{2}{5}t} + \frac{dy}{dt} + \frac{2}{25}y = 0$$

$$\frac{dy}{dt} + \frac{2}{25}y = 40e^{-\frac{2}{5}t}$$ (Inhomogeneous equation)

$$e^{\frac{2}{25}t}\frac{dy}{dt} + e^{\frac{2}{25}t}\frac{2}{25}y = e^{\frac{2}{25}t}\cdot 40e^{-\frac{2}{5}t} = 40e^{-\frac{8}{25}t}$$

$$\frac{d}{dt}\left(e^{\frac{2}{25}t}y\right) = 40e^{-\frac{8}{25}t}$$

$$e^{\frac{2}{25}t}y = -125e^{-\frac{8}{25}t} + C$$

Initial condition $y(0) = 0 \Rightarrow C = 125$

$$\therefore \quad y = 125e^{-\frac{2}{25}t}\left(1 - e^{-\frac{8}{25}t}\right)$$

$$= 125\left(e^{-\frac{2}{25}t} - e^{-\frac{2}{5}t}\right)$$

$$z = 100 - x - y$$

$$= 100 - 125e^{-\frac{2}{25}t} + 25e^{-\frac{2}{5}t}$$

c) Exercise Hint: $\frac{dy}{dt} = ?$

Ans: $t \approx 5.029$

e.g. Two kinds of bacteria, X and Y, coexist in environment. Both reproduce at a rate proportional to their numbers, and the constants of proportionality r_x and r_y $(r_y > r_x > 0)$ respectively. The environment provides sufficient resource so that no natural deaths occur during the time of investigation. However, each bacterium X kills bacteria Y at the rate of c per unit time. Initially, the population of X and Y are x_0 and kx_0 respectively.

Suppose that after time t, the population of X and Y are $x(t)$ and $y(t)$ respectively. Find $x(t)$ and $y(t)$.

$$\frac{dx}{dt} = r_x \cdot x \qquad \Rightarrow \qquad \frac{\frac{dx}{dt}}{x} = r_x$$

$$\int \frac{1}{x}\,dx = \int r_x\,dt$$

i.e. relative rate of change of $x(t)$ is a constant.

$$\ln x = r_x t + C_0$$

$$x = e^{r_x t + C_0} = C_0 e^{r_x t}$$

$$x(0) = x_0 \Rightarrow C_0 = x_0$$

$$\therefore x(t) = x_0 e^{r_x t}$$

(exponential growth)

$$\begin{aligned}\frac{dy}{dt} &= \text{rate of change of number of } Y \\ &= \text{rate of reproduction of } Y - \text{rate of death caused by killing from } X \\ &= r_y \cdot y - cx \\ &= r_y \cdot y - cx_0 e^{r_x t}\end{aligned}$$

$$\therefore \frac{dy}{dt} - r_y \cdot y = -cx_0 e^{r_x t}$$

(which is a 1st order linear ODE)

Integrating factor $= e^{-\int r_y dt} = e^{-r_y t}$

$$e^{-r_y t}\frac{dy}{dt} - e^{-r_y t} r_y \cdot y = -c x_0 e^{(r_x - r_y)t}$$

$$\frac{d}{dt}\left(e^{-r_y t} y\right) = -c x_0 e^{(r_x - r_y)t}$$

$$e^{-r_y t} y = -\int c x_0 e^{(r_x - r_y)t}\, dt$$

$$= -\frac{c x_0}{r_x - r_y} e^{(r_x - r_y)t} + C$$

$$y(0) = k x_0 \Rightarrow C = k x_0 + \frac{c x_0}{r_x - r_y}$$

$$\therefore\ e^{-r_y t} y = \frac{c x_0}{r_x - r_y} e^{(r_x - r_y)t} + k x_0 + \frac{c x_0}{r_x - r_y}$$

$$y = \frac{c x_0}{r_x - r_y} e^{r_x t} + \left(k x_0 + \frac{c x_0}{r_x - r_y}\right) e^{r_y t}$$

$$y = \frac{c x_0}{r_x - r_y} e^{r_x t} + \left(k + \frac{c}{r_x - r_y}\right) x_0 e^{r_y t}$$

If Y will extinct at some time t, i.e. $y(t) = 0$ for some $t > 0$.

That means $\frac{c x_0}{r_x - r_y} e^{r_x t} + \left(k + \frac{c}{r_x - r_y}\right) x_0 e^{r_y t} = 0$ has a solution.

$$\underbrace{\frac{c x_0}{r_x - r_y}}_{-ve} \underbrace{e^{r_x t}}_{+ve} = -\left(k + \frac{c}{r_x - r_y}\right) \underbrace{x_0 e^{r_y t}}_{+ve}$$

This equation has solution if and only if

$$-\left(k + \frac{c}{r_x - r_y}\right) < 0$$

$$-k < \frac{c}{r_x - r_y}$$

$$r_x > r_y - \frac{c}{k}$$

Note : $r_x < r_y \Rightarrow r_x - r_y < 0$

Explaination : We know $r_y > r_x > 0$, but if $r_x > r_y - \frac{c}{k}$, then Y will extinct !

(r_x is smaller that r_y but not so small)