MATH 1520C University Mathematics for Applications, 2014-15

Review :

1) Notations :

Set : collection of objects (elements)

$\subseteq$: subset

$\in$: belongs to

e.g. $S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means all elements in S and also in T.

Notations often used in this course :

$\mathbb{R}$: set of all real numbers

$[a, b]$: set of all real numbers x such that $a \leq x \leq b$

(a, b) : set of all real numbers x such that $a < x < b$

$[a, +\infty)$: set of all real numbers x such that $a \leq x$

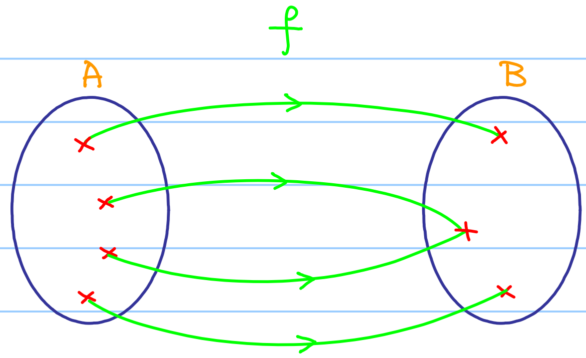
$\mathbb{R} \setminus \{a\}$: set of all real numbers except the number a

2) Functions :

Function : A function is a rule that assigns to each object in a set A exactly one object in a set B.

set A : domain (input)

set B : range (output)



A function f from A to B

We denote it by $f: A \to B$

e.g. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2+4$

$$\underset{\substack{\uparrow \\ \text{input}}}{f(-3)} = (-3)^2 + 4 = \underset{\substack{\uparrow \\ \text{output}}}{13}$$

OR write : $y = x^2+4$

dependent variable (y)   independent variable (x)

e.g. Find the domain of the functions :

a) $f(x) = \dfrac{1}{x-3}$

b) $g(t) = \dfrac{\sqrt{3-2t}}{t^2+4}$

Ans : a) x can be any real number except 3

i.e. domain $= \mathbb{R} \setminus \{3\}$

b) $3-2t \geq 0 \Rightarrow t \leq \frac{3}{2}$

i.e. domain $= (-\infty, \frac{3}{2}]$

e.g. What is the difference between $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x+1$?

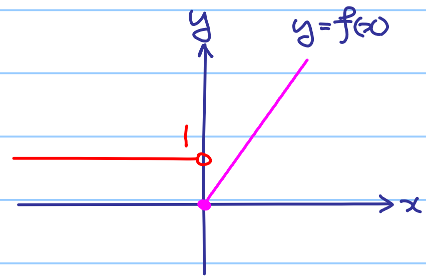
Ans : domain of $f = \mathbb{R} \setminus \{1\}$

domain of $g = \mathbb{R}$

e.g. Piecewise-defined function

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$



$f(-1) = 1$

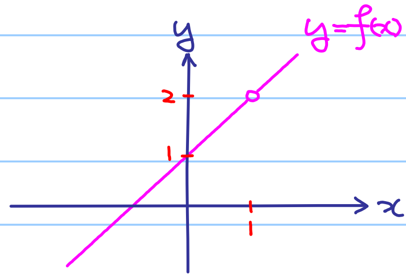
$f(0) = 0$

$f(1) = 2$

e.g. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



e.g. Composition of functions

$f, g : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = x^2 + 3x + 1$, $g(x) = x + 1$.

Think: $f(\triangle) = \triangle^2 + 3\triangle + 1$ $\quad$ $\triangle$: input

Then $f(g(x)) = (x+1)^2 + 3(x+1) + 1$ $\quad$ (Now: input $= \triangle = g(x) = x+1$)

$= x^2 + 5x + 5$

What is $g(f(x))$? $\quad$ Ans: $g(f(x)) = x^2 + 3x + 2$

Sometimes, we write $(f \circ g)(x)$ instead of $f(g(x))$ to emphasize it depends on x.

Functional Models :

Real World Problem :

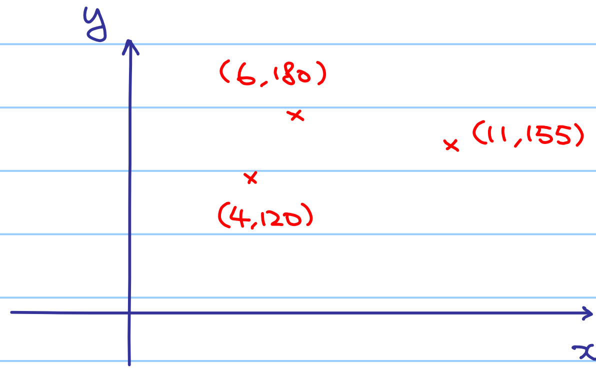
x : number of products produced (in thousands)

$P(x)$: profit (in thousands of dollars) (assumption : depending on x only)

Aim : Maximize the profit !

Step 1 : Observation

x	$P(x)$
4	120
6	180
11	155

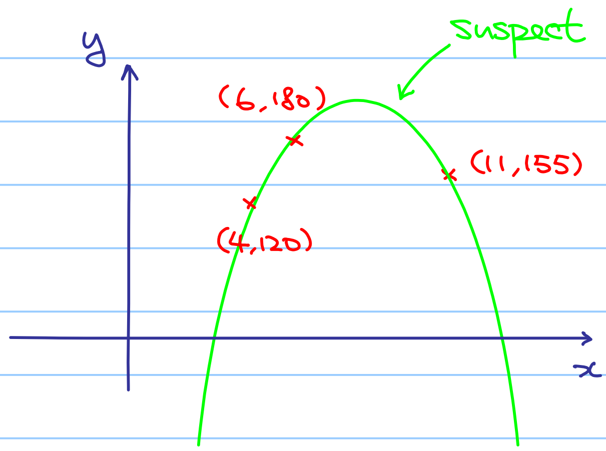


Step 2 : Modelling

Suspect the formula behind to be $P(x) = ax^2+bx+c$

Ex : Use the data above to solve a, b and c.

Ans : $a=-5$, $b=80$, $c=-120$



Step 3 : Prediction

Maximizing profit : $P(x) = -5x^2+80x-120$

$= -5(x-8)^2+200$

$\therefore$ Maximum profit $=200$ can be attained when $x=8$

Break-even : $P(x) = 0$

$-5x^2+80x-120 = 0$

$x = 8 \pm 2\sqrt{10}$

Step 4 : Testing the model

Accept , Modify or Reject ?

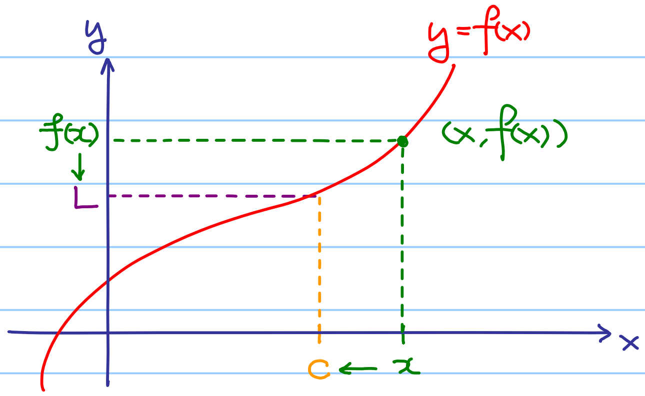
Question : How to find max / min for a general function ?

Answer : Calculus helps (also other applications)

Limits of Functions :

Limit of a function :

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer^{+} to c from both sides, then L is called the limit of $f(x)$ at c. We write $\lim_{x \to c} f(x) = L$

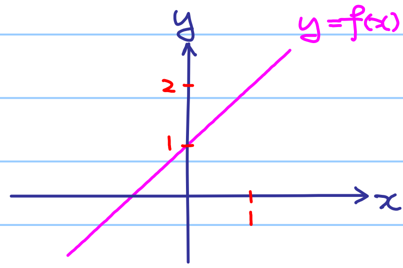


+ Note : a little bit misleading !

$f(c)$ may NOT equal to L, even it may be undefined !

e.g. If $f(x) = x+1$, find $\lim_{x \to 1} f(x)$

+ x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1



$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x \to 1} f(x) = 2$.

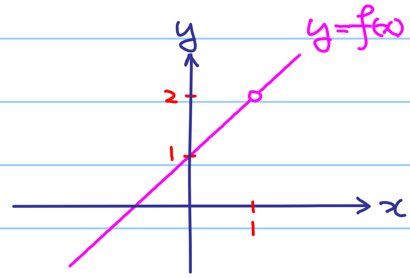
Remarks :

1) + The table only gives an intuitive idea, but NOT a rigorous proof !
2) Do NOT regard as putting $x=1$ into $f(x)$ and get $f(1)=2$!

e.g. Let $f: \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



graph of f

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1

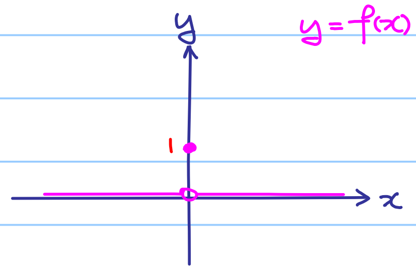
(But, we do NOT care what happens when $x = 1$!)

We write $\lim_{x \to 1} f(x) = 2$.

Compare with the previous example!

e.g. If $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$,

find $\lim_{x \to 0} f(x)$

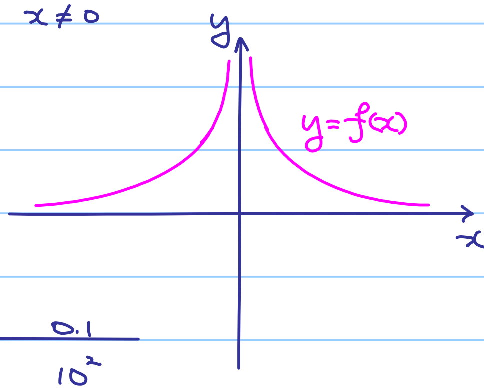


x	−0.1	−0.01	−0.001	$\boxed{0}$	0.001	0.01	0.1
$f(x)$	0	0	0	$\boxed{1}$	0	0	0

Do NOT care!

$\lim_{x \to 0} f(x) = 0$ which does NOT equal to $f(0) = 1$.

e.g. Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$, $x \neq 0$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0

$\therefore \lim\limits_{x \to 0} f(x)$ does NOT exist.

(But, some still write $\lim\limits_{x \to 0} f(x) = +\infty$.)

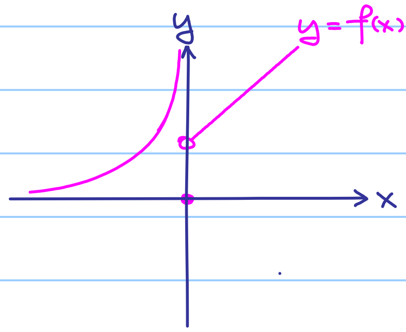
Right Hand Limit and Left Hand Limit:

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from right (resp. left) hand side, then L is called the right (resp. left) hand limit of $f(x)$ at c.

We write $\lim\limits_{x \to c^+} f(x) = L$ (resp. $\lim\limits_{x \to c^-} f(x) = L$)

e.g.

If $f(x) = \begin{cases} x+1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x < 0 \end{cases}$



$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x+1 = 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{1}{x^2}$ (does NOT exist)

$f(0) = 0$

Remark :

Right hand limit and left hand limit of a function at a point is NOT necessary to be the same !

FACT :

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

(i.e. both right and left hand limit exist and equal to L.)

FACT (Without proof)

(1) If k is a constant, $\lim_{x \to c} k = k$

(2) $\lim_{x \to c} x = c$ regarded as constant function $f(x) = k$

Algebraic Properties of Limits :

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist (very important !), then

(1) $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

(2) $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

(3) $\lim_{x \to c} (f(x) g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

(4) $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$

e.g. Find $\lim_{x\to 2} 3x^2-5$

① $\lim_{x\to 2} x = 2$, so $\lim_{x\to 2} x^2 = \lim_{x\to 2}(x\cdot x) \overset{\text{By (3)}}{=} \lim_{x\to 2} x \cdot \lim_{x\to 2} x$

$= 2\cdot 2 = 4$

② $\lim_{x\to 2} 3 = 3$, $\lim_{x\to 2} x^2 = 4$, so $\lim_{x\to 2} 3x^2 \overset{\text{By (3)}}{=} \lim_{x\to 2} 3 \cdot \lim_{x\to 2} x^2$

$= 3\cdot 4 = 12$

③ $\lim_{x\to 2} 3x^2 = 12$, $\lim_{x\to 2} 5 = 5$, so $\lim_{x\to 2} 3x^2-5 \overset{\text{By (2)}}{=} \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5$

$= 12-5 = 7$

But what we write:

$$\lim_{x\to 2} 3x^2-5 = 3\left(\lim_{x\to 2} x\right)^2 - 5 = 7$$

e.g. Find $\lim_{x\to 1} \frac{3x^2-8}{x-2}$

$$\lim_{x\to 1} \frac{3x^2-8}{x-2} = \frac{3\left(\lim_{x\to 1} x\right)^2 - 8}{\left(\lim_{x\to 1} x\right) - 2} = \frac{3(1)-8}{1-2} = 5$$

e.g. Think:

$$\lim_{x\to 0} \frac{1}{x} = \lim_{x\to 0} x\cdot\frac{1}{x^2} = \underbrace{\lim_{x\to 0} x}_{=\,0} \cdot \lim_{x\to 0}\frac{1}{x^2} = 0$$

(*)

But we know $\lim_{x\to 0} \frac{1}{x}$ does NOT exist.

What's wrong?

Ans: $\lim_{x\to 0} \frac{1}{x^2}$ does NOT exist, so we cannot use (3) at (*).

e.g. Find $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$

Note $\lim_{x\to 1} x^2-3x+2 = 0$, so we cannot use (4).

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x+1}{x-2} \overset{\text{By (4)}}{=} \frac{\lim_{x\to 1} x+1}{\lim_{x\to 1} x-2} = \frac{2}{-1} = -2$$

e.g. Let $f: \mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{\sqrt{x}-1}{x-1}$, $x \neq 1$.

Find $\lim_{x\to 1} f(x)$.

Note : For $x \neq 1$ ($x-1\neq 0$, denominator is nonzero.)

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1}$$

$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$ (We only concern those x near 1 but NOT equal to 1)

$= \frac{1}{2}$ (Still the same, do NOT regard as putting $x=1$)

Limit at Infinity :

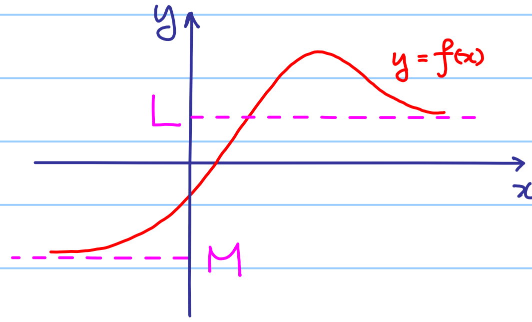
If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of $f(x)$ at $+\infty$.

We write $\lim\limits_{x\to+\infty} f(x) = L$.

(Similar definition for $\lim\limits_{x\to-\infty} f(x)$)

$\lim\limits_{x\to+\infty} f(x) = L$

$\lim\limits_{x\to-\infty} f(x) = M$



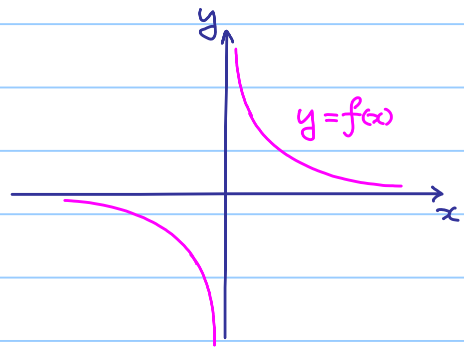
$\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are NOT necessary to be the same !

But if $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = L$, some simply write $\lim\limits_{x\to\infty} f(x) = L$.

e.g. $f(x) = \frac{1}{x}$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$

OR simply $\lim\limits_{x\to\infty} f(x) = 0$



FACT (without proof)

If $k > 0$, then $\lim\limits_{x\to\infty} \frac{1}{x^k} = 0$

Algebraic Properties of Limits at Infinity :

If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist (very important !) , then

(1) $\lim_{x\to+\infty} (f(x)+g(x)) = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x)$

(2) $\lim_{x\to+\infty} (f(x)-g(x)) = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x)$

(3) $\lim_{x\to+\infty} (f(x)g(x)) = \lim_{x\to+\infty} f(x) \cdot \lim_{x\to+\infty} g(x)$

(4) $\lim_{x\to+\infty} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ if $\lim_{x\to+\infty} g(x) \neq 0$

Similar results hold for limits at $-\infty$.

e.g. Find $\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1}$

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \neq \frac{\lim_{x\to+\infty} 3x^2}{\lim_{x\to+\infty} x^2+x+1}$$

Both limit do NOT exist !

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \frac{3}{1+0+0}$$

$$= 3$$

e.g. Find $\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$

$$\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{0+0}{3-0+0}$$

$$= 0$$

Compare the previous two examples :

For the second example,

$2x+1$ and $3x^2-2x+1$ tend to $+\infty$ as x tends to $+\infty$.

But $3x^2-2x+1$ grows "faster" than $2x+1$.

deg 2   deg 1

Think : If $p(x)$ and $q(x)$ are polynomials

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \quad \text{with } a_m > 0 \qquad (\text{i.e. } \deg p(x) = m)$$

$$q(x) = b_n x^n + a_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \quad \text{with } b_n > 0 \qquad (\text{i.e. } \deg q(x) = n)$$

then find $\displaystyle\lim_{x\to+\infty} \frac{p(x)}{q(x)}$ for the following cases :

1) $\deg p(x) > \deg q(x)$ i.e. $m > n$
2) $\deg p(x) = \deg q(x)$ i.e. $m = n$
3) $\deg p(x) < \deg q(x)$ i.e. $m < n$

Ans :

$$\lim_{x\to+\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty & \text{if } \deg p(x) > \deg q(x) \\ \dfrac{a_m}{b_m} & \text{if } \deg p(x) = \deg q(x) \\ 0 & \text{if } \deg p(x) < \deg q(x) \end{cases}$$