THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520C University Mathematics for Applications
Suggested Solution to Assignment 4

Ex 5.6: 11. Find the volume of the solid formed by rotating the region R about the x-axis. R is the region under the curve $y = \sqrt{4 - x^2}$ from $x = -2$ to $x = 2$

Solution: Since $y = \sqrt{4 - x^2}$ is an even function, the volume of the solid formed by rotating the region R about x-axis is

$$V = 2\int_0^2 \pi(\sqrt{4-x^2})^2\,dx = 2\pi\left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \frac{32\pi}{3}$$

Ex 5.6: 31 **GROWTH OF AN ENDANGERED SPECIES** Environmentalists estimate that the population of a certain endangered species is currently 3,000. The population is expected to be growing at the rate of $R(t) = 10e^{0.01t}$ individuals per year t years from now, and the fraction that survive t years is given by $S(t) = e^{-0.07t}$. What will the population of the species be in 10 years?

Solution: To approximate the number of new individuals which will survive 10 years from now, divide the interval $0 \leq t \leq 10$ into n equal subintervals of length Δt. Moreover $t_j = j \times \Delta t$ and the j-th interval is $[t_j, t_{j+1}]$. The new individuals grown in the j-th interval is $R(t_j)S(10 - t_j)\Delta t$. Therefore the new individuals grown in the 10 years can be approximated by

$$\sum_{j=1}^{n} 10e^{0.01t_j}e^{-0.07(10-t_j)}\Delta t$$

which implies the population of the species in 10 years can be approximated by

$$P \approx 3000S(10) + \sum_{j=1}^{n} 10e^{0.01t_j}e^{-0.07(10-t_j)}\Delta t$$

Therefore

$$\begin{aligned}P(10) &= 3000S(10) + 10e^{-0.7}\int_0^{10} e^{0.08t}\,dt\\ &= 3000e^{-0.7} + 10e^{-0.7}\left(\frac{e^{0.08t}}{0.08}\right)\Big|_0^{10}\\ &\approx 1565.83(1566)\end{aligned}$$

Since the number of species should be integer, the population should be 1566 individuals.

Ex 6.1: 35. Use the table of integrals (Table 6.1) to find the integral of $\int (\ln x)^3\,dx$.

Solution:

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3\int (\ln x)^2\,dx \\ &= x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\int \ln x\,dx\right) \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6\left(x\ln x - \int 1\,dx\right) \\ &= x(\ln x)^3 - 3(\ln x)^2 + 6x\ln x - 6x + C\end{aligned}$$

Ex 9.1: 25. Find the particular solution of the differential equation satisfying the indicated condition.

$$\frac{dy}{dx} = y^2\sqrt{4-x};\, y = 2 \text{ when x} = 4$$

Solution: Since $\frac{dy}{dx} = y^2\sqrt{4-x}$,

$$\int \frac{1}{y^2}\,dy = \int \sqrt{4-x}\,dx$$

That is

$$-\frac{1}{y} = -\frac{2}{3}(4-x)^{\frac{3}{2}} + C$$

Since $y = 2$ when $x = 4$

$$\frac{1}{2} = C$$

Therefore the particular solution is

$$y = \frac{1}{\frac{2}{3}(4-x)^{\frac{3}{2}} + \frac{1}{2}} = \frac{6}{4(4-x)^{\frac{3}{2}} + 3}$$

Ex 9.2: 5,7. In Exercise 5 and 7, find the general solution of the given first-order linear differential equation.

5. $x^2\frac{dy}{dx} + xy = 2$

Solution: This differential equation can be rewritten as $\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x^2}$. Therefore the integral factor is

$$e^{\int x^{-x}\,dx} = x$$

The general solution is

$$y = \frac{1}{x}\left[\int x \cdot \frac{2}{x^2}\,dx + C\right] = \frac{2\ln|x| + C}{x}$$

7. $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right) y = e^{-2x}$

Solution: The integral factor is $I(x) = e^{\int \frac{2x+1}{x}\,dx} = xe^{2x}$. The general solution is

$$y = \frac{1}{xe^{2x}}\left[\int xe^{2x} \cdot e^{-2x}\,dx + C\right] = \frac{x^2 + 2C}{2xe^{2x}}$$

Note that $2C$ or C are of same meaning, so both of them are right.