THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520C University Mathematics for Applications
Suggested Solution to Assignment 2

Exercise 2.4: 81. The growth of certain insects varies with temperature. Suppose a particular species of insect grows in such a way that the volume of an individual is $V(T) = 0.41(-0.01T^2 + 0.4T + 3.52)\text{cm}^3$ when the temperature is $T°C$, and that its mass is m grams, where

$$m(V) = \frac{0.39V}{1 + 0.09V}$$

(a) Find the rate of change of the insect's volume with respect to temperature.
(b) Find the rate of the insect's mass with respect to volume.
(c) When T=10°C, what is the insect's volume? At what rate is the insect's mass changing with respect to temperature when T=10°C?

Solution: (a) $V'(T) = 0.41(-0.02T + 0.4) = -0.0082T + 0.164$
(b) $m'(V) = \frac{(1+0.09V)(0.39)-(0.39V)(0.09)}{(1+0.09V)^2} = \frac{0.39}{(1+0.09V)^2}$
(c) When T=10°C, V(10) = 2.6732 cm^3
The changing rate of the insect's mass with respect to temperature when T=10°C
$\frac{dm}{dT}(10) = m'(V(10))V'(10) = \frac{0.39}{(1+0.09(2.6732))^2}(-0.0082(10)+0.164) \approx 0.0208 \text{ g}°C^{-1}$.

Exercise 4.3: Differentiate the given functions.

25. $h(t) = \frac{e^t+t}{\ln t}$.

Solution:

$$h'(t) = \frac{((e^t + 1)\ln t - (e^t + t)\frac{1}{t}}{(\ln t)^2} = \frac{te^t \ln t + t\ln t - e^t - t}{t(\ln t)^2}$$

29. $f(t) = \sqrt{\ln t + t}$.

Solution:

$$f'(t) = \frac{\frac{1}{t} + 1}{2\sqrt{\ln t + t}} = \frac{t + 1}{2t\sqrt{\ln t + t}}.$$

37. $f(x) = x \log_{10} x$.

Solution:

$$f'(x) = \log_{10} x + x\frac{1}{x\ln 10} = \log_{10} x + \frac{1}{\ln 10}.$$

61,63 Use logarithmic differentiation to find the derivative $f'(x)$.

61. $f(x) = (x+1)^3(6-x)^2\sqrt[3]{2x+1}$.

Solution: Since

$$\ln f(x) = 3\ln(x+1) + 2\ln(6-x) + \frac{1}{3}\ln(2x+1),$$
$$\frac{f'(x)}{f(x)} = \frac{3}{x+1} - \frac{2}{6-x} + \frac{2}{3(2x+1)}$$
$$\Longrightarrow f'(x) = f(x)\left(\frac{3}{x+1} - \frac{2}{6-x} + \frac{2}{3(2x+1)}\right)$$

63. $f(x) = 5^{x^2}$.

Solution: Since

$$\ln f(x) = x^2\ln 5$$
$$\frac{f'(x)}{f(x)} = 2x\ln 5$$
$$\Longrightarrow f'(x) = 2xf(x)\ln 5 = 2x5^{x^2}\ln 5$$

85. An international agency determines the number of individuals of an endangered species that remain in the wild t years after a protection policy is instituted may be modeled by

$$N(t) = \frac{600}{1+3e^{-0.02t}}$$

(a) At what is the population changing at time t? When is the population increasing? When is it decreasing?
(b) When is the rate of change of the population increasing? When is it decreasing? Interpret your results.
(c) What happens to the population in the long run(as $t \to +\infty$)?

Solution: (a) $N'(t) = \frac{36e^{-0.02t}}{(1+3e^{-0.02t})^2} > 0$. The population is increasing at all times t.
(b) Since $N''(t) = -0.72\frac{e^{-0.02t}}{(1+3e^{-0.02t})^2} + (-2)\frac{(-0.06)e^{-0.02t}36e^{-0.02t}}{(1+3e^{-0.02t})^3} = \frac{e^{-0.02t}(2.16e^{-0.02t}-0.72)}{(1+3e^{-0.02t})^3}$.
Letting $N''(t) > 0$, it is equivalent to

$$2.16e^{-0.02t} - 0.72 > 0 \quad \Longrightarrow t < 50\ln 3$$

Therefore the rate of change of the population is increasing when $t < 50\ln 3$, decreasing when $t > 50\ln 3$.

(c) $\lim_{t\to+\infty} N(t) = 600$.

1 (a) Since $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x-1) = 1$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} x^2 = 1$, $\lim_{x\to 1} f(x) = 1 = f(1)$. Therefore $f(x)$ is continuous at $x = 1$.

(b) Note that

$$f'_+(1) = \lim_{\Delta x\to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \frac{2\Delta x}{\Delta x} = 2$$

$$f'_-(1) = \lim_{\Delta x\to 0} \frac{f(1) - f(1-\Delta x)}{\Delta x} = \frac{2\Delta x - (\Delta x)^2}{\Delta x} = 2$$

Therefore $f'(1)$ exists and $f'(1) = 2$.

2 (a)

$$f'(x) = \frac{3x^2}{x^2-1} - \frac{x^3(2x)}{(x^2-1)^2} = \frac{x^4-3x^2}{(x^2-1)^2}$$

$$f''(x) = \frac{4x^3-6x}{(x^2-1)^2} - 2\frac{(x^4-3x^2)(2x)}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3}$$

(b) Fist, note that $x \neq \pm 1$ by the definition.

(i) $f'(x) > 0$ when $x^4 - 3x^2 = x^2(x^2-3) > 0$, that is $x < -\sqrt{3}$ or $x > \sqrt{3}$.

(ii) $f'(x) < 0$ when $-\sqrt{3} < x < \sqrt{3}$ and $x \neq \pm 1, 0$.

(iii) $f''(x) > 0$ when $\frac{2x(x^2+3)}{(x^2-1)^3} > 0$

$$\Longleftrightarrow x(x^2-1)(x^2-1)^2 > 0$$
$$\Longleftrightarrow x(x^2-1) > 0$$
$$\longrightarrow -1 < x < 0 \quad \text{or} \quad x > 1$$

(iv) $f''(x) < 0$ when $x < -1$ or $0 < x < 1$.

(c) Saddle/max/min point(s), point(s) of inflection:

x	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
$f'(x)$	+	0	-	0	-	0	+
$f''(x)$	-	-	-	0	+	+	+

The local maximum point is $x = -\sqrt{3}$, the local minimum point is $x = \sqrt{3}$. The point of inflection is $x = 0$.

(d) horizontal/vertical asymptote(s):

As $\lim_{x\to\pm 1} f(x) = \lim_{x\to\pm 1} \frac{x^3}{x^2-1} = \infty$, $x = \pm 1$ are the vertical asymptotes.
(e)