THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520C University Mathematics for Applications 2014-2015

Revision

Note: Questions will be discussed in lectures, no typed solution will be given.

1. Evaluate the following limits.
   (a) $\lim_{x\to+\infty} \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$.
   (b) $\lim_{x\to+\infty} (\sqrt{x+1} - \sqrt{x})\sqrt{x+2}$.
   (c) $\lim_{x\to+\infty} (\sqrt{x^2+1} - x)$.
   (d) $\lim_{x\to 0} \dfrac{(1+2x)(1+3x)(1+4x) - 1}{x}$.

2. By using the fact that $\lim_{x\to-\infty} \left(1+\dfrac{1}{x}\right)^x = \lim_{x\to+\infty} \left(1+\dfrac{1}{x}\right)^x = \lim_{y\to 0}(1+y)^{1/y} = e$, compute the following limits.
   (a) $\lim_{x\to+\infty} \left(1-\dfrac{1}{x}\right)^{-x}$.
   (b) $\lim_{x\to+\infty} \left(1+\dfrac{3}{x^2}\right)^{x}$.
   (c) $\lim_{x\to-\infty} \left(\dfrac{x^2-2x-3}{x^2-3x-28}\right)^{x}$.
   (d) $\lim_{x\to 0}(1-3x)^{1/x}$.

3. Find the derivatives of the following functions.
   (a) $f(x) = \sqrt{x^2 - x + 1}$
   (b) $f(x) = 4^{\frac{x}{\ln x}}$
   (c) $f(x) = \log_5(x^2 + 3x - 1)$

4. Evaluate the following integrals.
   (a) $\int_{-2}^{4} |x^2 - 3x + 2| \, dx$
   (b) $\int_0^{\ln 2} \dfrac{(e^x - e^{-x})^2}{e^x} \, dx$
   (c) $\int_1^{e} \dfrac{\ln x}{x^2} \, dx$

5. (a) If $u = e^x + e^{-x}$, find $\dfrac{du}{dx}$.
   (b) Hence, evaluate $\int_0^{\ln 2} \dfrac{e^{2x} - 1}{e^{2x} + 1} \, dx$.

6. Jules decides to go on a diet for 6 weeks, with a goal of losing between 10 and 15 pounds. Based on his body configuration and metabolism, his doctor determines that the amount of weight he will lose can be modeled by a continuous random variable X with probability density function $f(x)$ of the form

$$f(x) = \begin{cases} k(x-10)^2 & \text{for } 10 \leq x \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

If the doctor's model is valid, how much weight should Jules expect to lose?

7. Brooke, the manager of a fishery, determines that the age X (in weeks) at which a certain species of fish dies follows an exponential distribution with probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Brooke observes that it is twice as likely for a randomly selected fish to die during the first 10-week period as during the next 10 weeks (from week 10 to week 20).

   (a) What is λ?

   (b) What is the probability that a randomly chosen fish will die within the first 5 weeks?

   (c) How long should Brooke expect a randomly selected fish to live?

8. Let X be a random variable that is distributed with a probability density function of the form

$$f(x) = \begin{cases} axe^{-bx} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are positive real numbers.

   (a) Prove that $a = b^2$.

   (b) Find the expected value and variance of X.

   (c) What is the range of b if the variance of X is less than 1?

9. The differential equation

$$\frac{dQ}{dt} = Q(a - b\ln Q)$$

where a and b are positive real numbers, $Q(t) > 0$, is called the Gompertz equation, and a solution of the equation is called a Gompertz function. Such functions are used to describe restricted growth in populations as well as matters such as learning and growth within an organization.

   (a) Use the Gompertz equation to show that a Gompertz function is growing most rapidly when $\ln Q = \dfrac{a-b}{b}$.

   (b) Solve the Gompertz equation.

   (c) Compute $\lim\limits_{t \to +\infty} Q(t)$.

   (d) Sketch the graph of a typical Gompertz function.

10. In a certain chain reaction

$$X \longrightarrow Y \longrightarrow Z,$$

radioactive element X decays into radioactive element Y which in turn decays into element Z. The number of atoms of X, Y and Z at time t are x, y and z respectively. The total number of atoms, $x + y + z$, is constant over time. The rates of decay of X and Y are $k_1 x$ and $k_2 y$ respectively ($k_2 > k_1 > 0$). At time $t = 0$, $x = A$ and $y = z = 0$.

(a) Show that

(i) $x = Ae^{-k_1 t}$, and

(ii) $y = \dfrac{Ak_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t})$.

Hence deduce the value of z at time t.

(b) Find the time at which y attains its maximum. What is the maximum value?

11. After t years of operation, a certain nuclear power plant produces radioactive waste at the rate $R(t) = 300 - 200e^{-0.03t}$ pounds per year. The waste decays exponentially at the rate of 2% per year. How much radioactive waste from the plant will be present in the long run?