

THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520C University Mathematics for Applications 2014-2015

Revision

Note: Questions will be discussed in lectures, no typed solution will be given.

1. Evaluate the following limits.

(a) $\lim_{x \rightarrow +\infty} \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$.

(b) $\lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x})\sqrt{x+2}$.

(c) $\lim_{x \rightarrow +\infty} (\sqrt{x^2+1} - x)$.

(d) $\lim_{x \rightarrow 0} \frac{(1+2x)(1+3x)(1+4x) - 1}{x}$.

2. By using the fact that $\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{y \rightarrow 0} (1+y)^{1/y} = e$, compute the following limits.

(a) $\lim_{x \rightarrow +\infty} \left(1 - \frac{1}{x}\right)^{-x}$.

(b) $\lim_{x \rightarrow +\infty} \left(1 + \frac{3}{x^2}\right)^x$.

(c) $\lim_{x \rightarrow -\infty} \left(\frac{x^2 - 2x - 3}{x^2 - 3x - 28}\right)^x$.

(d) $\lim_{x \rightarrow 0} (1 - 3x)^{1/x}$.

3. Find the derivatives of the following functions.

(a) $f(x) = \sqrt{x^2 - x + 1}$

(b) $f(x) = 4^{\frac{x}{\ln x}}$

(c) $f(x) = \log_5(x^2 + 3x - 1)$

4. Evaluate the following integrals.

(a) $\int_{-2}^4 |x^2 - 3x + 2| dx$

(b) $\int_0^{\ln 2} \frac{(e^x - e^{-x})^2}{e^x} dx$

(c) $\int_1^e \frac{\ln x}{x^2} dx$

5. (a) If $u = e^x + e^{-x}$, find $\frac{du}{dx}$.

(b) Hence, evaluate $\int_0^{\ln 2} \frac{e^{2x} - 1}{e^{2x} + 1} dx$.

6. Jules decides to go on a diet for 6 weeks, with a goal of losing between 10 and 15 pounds. Based on his body configuration and metabolism, his doctor determines that the amount of weight he will lose can be modeled by a continuous random variable X with probability density function $f(x)$ of the form

$$f(x) = \begin{cases} k(x-10)^2 & \text{for } 10 \leq x \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

If the doctor's model is valid, how much weight should Jules expect to lose?

7. Brooke, the manager of a fishery, determines that the age X (in weeks) at which a certain species of fish dies follows an exponential distribution with probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Brooke observes that it is twice as likely for a randomly selected fish to die during the first 10-week period as during the next 10 weeks (from week 10 to week 20).

- What is λ ?
 - What is the probability that a randomly chosen fish will die within the first 5 weeks?
 - How long should Brooke expect a randomly selected fish to live?
8. Let X be a random variable that is distributed with a probability density function of the form

$$f(x) = \begin{cases} axe^{-bx} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are positive real numbers.

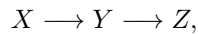
- Prove that $a = b^2$.
 - Find the expected value and variance of X .
 - What is the range of b if the variance of X is less than 1?
9. The differential equation

$$\frac{dQ}{dt} = Q(a - b \ln Q)$$

where a and b are positive real numbers, $Q(t) > 0$, is called the Gompertz equation, and a solution of the equation is called a Gompertz function. Such functions are used to describe restricted growth in populations as well as matters such as learning and growth within an organization.

- Use the Gompertz equation to show that a Gompertz function is growing most rapidly when $\ln Q = \frac{a-b}{b}$.
- Solve the Gompertz equation.
- Compute $\lim_{t \rightarrow +\infty} Q(t)$.
- Sketch the graph of a typical Gompertz function.

10. In a certain chain reaction



radioactive element X decays into radioactive element Y which in turn decays into element Z . The number of atoms of X , Y and Z at time t are x , y and z respectively. The total number of atoms, $x + y + z$, is constant over time. The rates of decay of X and Y are k_1x and k_2y respectively ($k_2 > k_1 > 0$). At time $t = 0$, $x = A$ and $y = z = 0$.

(a) Show that

(i) $x = Ae^{-k_1t}$, and

(ii) $y = \frac{Ak_1}{k_2 - k_1}(e^{-k_1t} - e^{-k_2t})$.

Hence deduce the value of z at time t .

(b) Find the time at which y attains its maximum. What is the maximum value?

11. After t years of operation, a certain nuclear power plant produces radioactive waste at the rate $R(t) = 300 - 200e^{-0.03t}$ pounds per year. The waste decays exponentially at the rate of 2% per year. How much radioactive waste from the plant will be present in the long run?