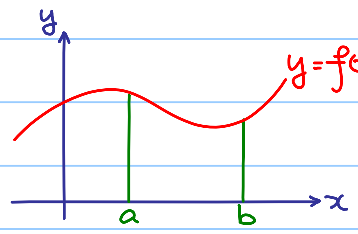
Inequalities Involving Integrals

Theorem : If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $f(x)\geq 0$ for all $x\in[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.



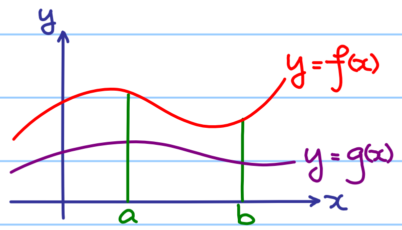
Corollary : If $f,g:[a,b]\to\mathbb{R}$ are continuous functions such that $f(x)\geq g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

proof : $f(x)-g(x)\geq 0$ for all $x\in[a,b]$

$$\Rightarrow \int_a^b f(x)-g(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$



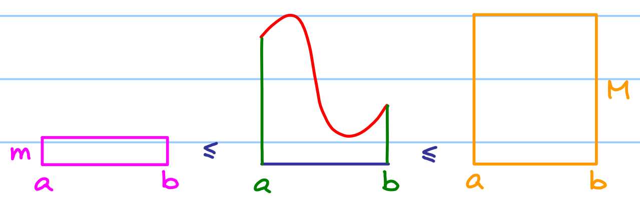
Corollary : If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $m\leq f(x)\leq M$ for all $x\in[a,b]$, where $m,M\in\mathbb{R}$, then

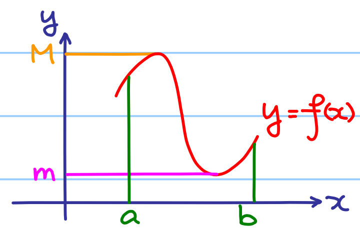
$$m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a)$$

proof : $m\leq f(x)\leq M$ for all $x\in[a,b]$

$$\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$\Rightarrow m(b-a)\leq \int_a^b f(x)\,dx \leq M(b-a)$$





e.g. Define $a_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n \in \mathbb{N}$, show that $\frac{1}{2(n+1)} \le a_n \le \frac{1}{n+1}$.

Hence, find $\lim_{n\to\infty} a_n$.

Note : For $0 \le x \le 1$,

$$1 \le 1+x^2 \le 2$$

$$\frac{1}{2} \le \frac{1}{1+x^2} \le 1$$

$$\frac{x^n}{2} \le \frac{x^n}{1+x^2} \le x^n$$

$$\int_0^1 \frac{x^n}{2}\,dx \le \int_0^1 \frac{x^n}{1+x^2}\,dx \le \int_0^1 x^n\,dx$$

$$\frac{1}{2(n+1)} \le \int_0^1 \frac{x^n}{1+x^2}\,dx \le \frac{1}{n+1}$$

$\because \lim_{n\to\infty} \frac{1}{2(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x^2}\,dx = 0$

e.g. Let $I_n = \int_0^1 e^t t^n\,dt$ where n is nonnegative integer.

a) Prove that $I_n = e - nI_{n-1}$ for $n \ge 1$.

Hence, deduce that $I_n = (-1)^{n+1} n! + e\sum_{r=0}^{n} (-1)^i \frac{n!}{(n-r)!}$.

b) Show that $\frac{1}{n+1} \le I_n \le \int_0^1 et^n\,dt < \frac{e}{n}$ for all $n \ge 1$.

c) Hence, prove that e is an irrational number.

a) For $n \ge 1$, $I_n = \int_0^1 e^t t^n\,dt$

$$= \int_0^1 t^n\,de^t$$

$$= [t^n e^t]_0^1 - \int_0^1 e^t\,dt^n$$

$$= e - \int_0^1 ne^t t^{n-1}\,dt$$

$$= e - nI_{n-1}$$

$$I_n = e - nI_{n-1}$$
$$= e - n[e-(n-1)I_{n-2}]$$
$$= e - ne + n(n-1)I_{n-2}$$
$$= e - ne + n(n-1)[e-(n-2)I_{n-3}]$$
$$= e - ne + n(n-1)e - n(n-1)(n-2)I_{n-3}$$
$$\vdots$$
$$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot I_0$$
$$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot (e-1)$$
$$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot e$$
$$+(-1)^{n+1}n(n-1)(n-2)\cdots 2\cdot 1$$
$$= (-1)^{n+1}n! + e\sum_{r=0}^{n}(-1)^i\frac{n!}{(n-r)!}$$

$$I_0 = \int_0^1 e^t\,dt$$
$$= [e^t]_0^1$$
$$= e-1$$

b) Note: For $0\le t\le 1$, $t^n \le e^t t^n \le et^n$

$$\int_0^1 t^n\,dt \le \underbrace{\int_0^1 e^t t^n\,dt}_{=I_n} \le \int_0^1 et^n\,dt$$

$$\int_0^1 t^n\,dt = \left[\frac{1}{n+1}t^{n+1}\right]_0^1 = \frac{1}{n+1}$$
$$\int_0^1 et^n\,dt = e\int_0^1 t^n\,dt = \frac{e}{n+1} < \frac{e}{n}$$
$$\therefore \frac{1}{n+1} \le I_n \le \int_0^1 et^n\,dt < \frac{e}{n}$$

c) From (a) and (b),

$$\frac{1}{n+1} \le I_n < \frac{e}{n}$$
$$\frac{1}{n+1} \le (-1)^{n+1}n! + e\sum_{r=0}^{n}(-1)^i\frac{n!}{(n-r)!} < \frac{e}{n}$$
$$\frac{1}{e(n+1)} \le \frac{(-1)^{n+1}n!}{e} + \sum_{r=0}^{n}(-1)^i\frac{n!}{(n-r)!} < \frac{1}{n}$$

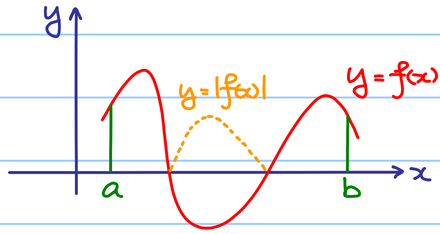
If e is rational, when we consider a sufficiently large n,
then $\frac{(-1)^{n+1}n!}{e} + \sum_{r=0}^{n}(-1)^i\frac{n!}{(n-r)!}$ is an integer which is impossible as $0 < \frac{1}{e(n+1)}$ and $\frac{1}{n} < 1$!

$\therefore$ e is irrational.

Corollary: If $f:[a,b]\to\mathbb{R}$ is a continuous function, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

(i.e. $-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$)



proof: Note: $-|f(x)| \le f(x) \le |f(x)|$

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

e.g. Let $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx$, for $n\in\mathbb{N}$, prove that $|I_n| \le \frac{\pi}{4n}$.

Hence, deduce $\lim\limits_{n\to\infty} I_n = 0$.

$$|I_n| = \frac{1}{n}\left|\int_0^1 \frac{\sin nx}{1+x^2}\,dx\right|$$

$$\le \frac{1}{n}\int_0^1 \left|\frac{\sin nx}{1+x^2}\right|dx$$

$$\le \frac{1}{n}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{n}\left[\tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{4n}$$

$0 \le |I_n| \le \frac{\pi}{4n}$ and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty}\frac{\pi}{4n} = 0$

$\therefore$ By sandwich theorem, $\lim\limits_{n\to\infty}|I_n| = 0$ and so $\lim\limits_{n\to\infty} I_n = 0$.