

Indefinite Integral:

Antiderivative: A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

e.g. If $f(x) = 2x$, $F(x) = x^2$,

then we have $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x) = x^2 + C$, where C is a constant.

Then, we still have $F'(x) = f(x)$.

Therefore, antiderivative of a function $f(x)$ is NOT unique.

That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,
what is the relation between them?

Answer: $F(x)$ and $G(x)$ differ by a constant.

proof: Suppose $F'(x) = G'(x) = f(x)$

Let $H(x) = F(x) - G(x)$

Then $H'(x) = F'(x) - G'(x) = 0$

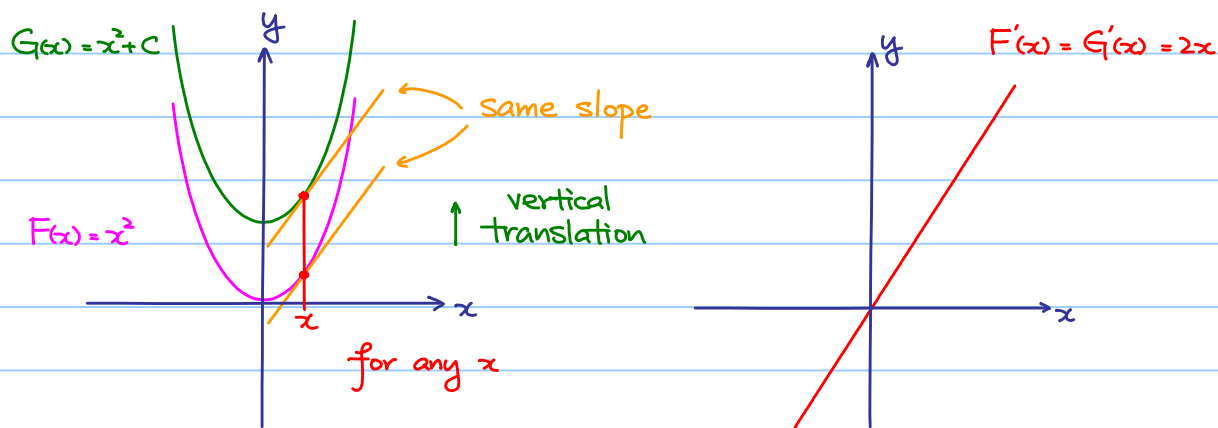
$\therefore H(x)$ is a constant function, i.e. $H(x) = C$ for some constant C .

i.e. $F(x) = G(x) + C$

Therefore, antiderivative of a function $f(x)$ is NOT unique,
but it is unique up to a constant.

e.g. If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.



If $F(x)$ is an antiderivative of $f(x)$, we write

$$\int f(x) dx = F(x) + C$$

Annotations: 'integrand' points to $f(x)$, 'integral symbol' points to \int , and 'variable of integration' points to dx .

e.g. $\int 2x dx = x^2 + C$

If $F(x)$ is an antiderivative of $f(x)$,

$$F(x) \xrightarrow{\text{differentiate}} f(x) \xrightarrow{\text{integrate}} \int f(x) dx = F(x) + C$$

= original function up to a constant

Note: When we write $\int f(x) dx$, sometimes it may be regarded as a class of functions.

Rules for Integrating Common Functions

1) $\int k dx = kx + C$, for constant k .

2) $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$, for all n except -1 .

3) $\int \frac{1}{x} dx = \ln|x| + C$

4) $\int e^x dx = e^x + C$

5) $\int \cos x dx = \sin x + C$

6) $\int \sin x dx = -\cos x + C$

7) $\int \frac{1}{1+x^2} dx = \tan^{-1}x + C$

Algebraic Rules For Indefinite Integration

1) $\int k f(x) dx = k \int f(x) dx$

2) $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

Note: 1) $\frac{d}{dx} (\int k f(x) dx) = \frac{d}{dx} (k \int f(x) dx) = k f(x)$

i.e. $\int k f(x) dx$ and $k \int f(x) dx$ differ by a constant.
but it is absorbed by \int .

2) $\frac{d}{dx} (\int f(x) \pm g(x) dx) = \frac{d}{dx} (\int f(x) dx \pm \int g(x) dx) = f(x) \pm g(x)$.

e.g. $\int 2x^5 - 3x^2 + 7x + 5 \, dx$

$= 2 \int x^5 \, dx - 3 \int x^2 \, dx + 7 \int x \, dx + 5 \int dx$

$\int dx$ means $\int 1 \, dx$

\int still there.

No need to add $+C$!

$= 2 \left(\frac{x^6}{6} \right) - 3 \left(\frac{x^3}{3} \right) + 7 \left(\frac{x^2}{2} \right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

e.g. $\int \frac{x^3 - 5}{x} \, dx$

$= \int x^2 - \frac{5}{x} \, dx$

$= \frac{x^3}{3} - 5 \ln|x| + C$

e.g. Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

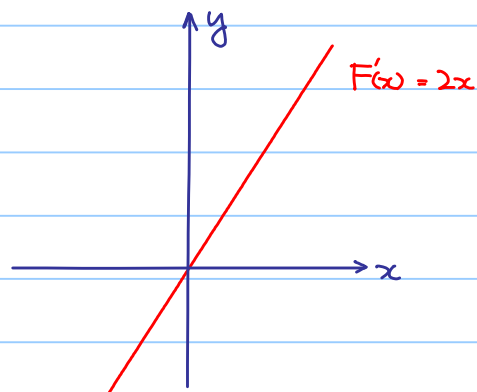
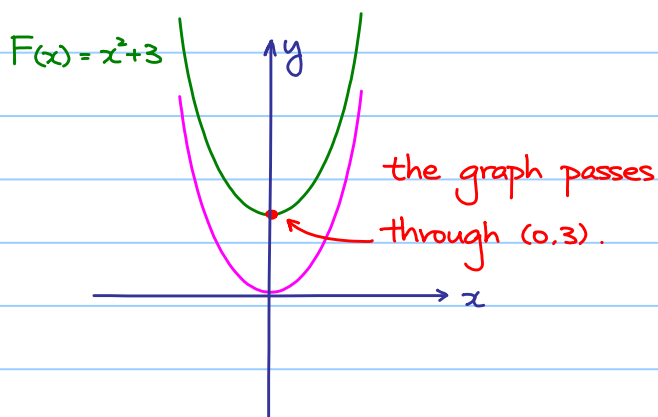
$F'(x) = 2x$

$F(x) = \int 2x \, dx$

$= x^2 + C$

$F(0) = 0^2 + C = 3 \Rightarrow C = 3$

$\therefore F(x) = x^2 + 3$



Integration by Substitution

Question : $\int (2x+1)^{2015} dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Integration by Substitution : $\int f(u(x)) u'(x) dx = \int f(u) du$

$$\text{OR : } \int f(u) \frac{du}{dx} dx = \int f(u) du$$

proof : $\frac{d}{dx} \int f(u(x)) u'(x) dx = f(u(x)) u'(x)$

$$\frac{d}{dx} \int f(u) du = \frac{d}{du} \int f(u) du \cdot \frac{du}{dx} \quad (\text{Chain Rule})$$

$$= f(u(x)) \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \int f(u(x)) u'(x) dx = \frac{d}{dx} \int f(u) du$$

$$\therefore \int f(u(x)) u'(x) dx = \int f(u) du$$

e.g. $\int (2x+1)^{2015} dx = ?$

$$\text{Let } u(x) = 2x+1 \quad u'(x) = 2$$

$$f(u) = u^{2015} \quad f(u(x)) = (2x+1)^{2015}$$

$$\int (2x+1)^{2015} dx = \frac{1}{2} \int \underbrace{(2x+1)^{2015}}_{f(u(x))} \cdot \underbrace{2}_{u'(x)} dx = \frac{1}{2} \int \underbrace{u^{2015}}_{f(u)} du$$

$$= \frac{1}{4032} u^{2016} + C = \frac{1}{4032} (2x+1)^{2016} + C$$

But, usually we write,

$$\int (2x+1)^{2015} dx$$

$$= \int u^{2015} \frac{1}{2} du$$

$$= \frac{1}{4032} u^{2016} + C$$

$$= \frac{1}{4032} (2x+1)^{2016} + C$$

$$\text{Let } u = 2x+1$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{1}{2} du$$

$$\begin{aligned} \text{e.g. } \int e^{ax} dx &= \int e^u \cdot \frac{1}{a} du \\ &= \frac{1}{a} e^u + C \\ &= \frac{1}{a} e^{ax} + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= ax \\ \frac{du}{dx} &= a \\ dx &= \frac{1}{a} du \end{aligned}$$

$$\begin{aligned} \text{e.g. } \int 6x(4x^2+3)^7 dx &= \int 6(4x^2+3)^7 x dx \\ &= \int 6 u^7 \frac{1}{8} dx \\ &= \frac{6}{8} \cdot \frac{1}{8} u^8 + C \\ &= \frac{3}{32} (4x^2+3)^8 + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= 4x^2+3 \\ \frac{du}{dx} &= 8x \\ x dx &= \frac{1}{8} du \end{aligned}$$

$$\text{e.g. } \int \frac{(\ln x)^2}{x} dx, \quad x > 0$$

$$\begin{aligned} \int \frac{(\ln x)^2}{x} dx &= \int u^2 du \\ &= \frac{1}{3} u^3 + C \\ &= \frac{1}{3} (\ln x)^3 + C \end{aligned}$$

$$\begin{aligned} \text{Let } u &= \ln x \\ \frac{du}{dx} &= \frac{1}{x} \\ \frac{1}{x} dx &= du \end{aligned}$$

Question: How to make a guess of $u(x)$?

Integration by Substitution: $\int f(u(x)) u'(x) dx = \int f(u) du$

$$\text{e.g. } \int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \cdot \frac{1}{x} dx \quad \text{Let } u = \ln x$$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$

Ex: 1) Show that $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$. Hint: Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4} dx$

Hint: Let $u = x^4$

Ans: $\frac{1}{4} e^{x^4} + C$

b) $\int 6x \sqrt{x^2+3} dx$

Hint: Let $u = x^2+3$

Ans: $2(x^2+3)^{\frac{3}{2}} + C$

Integration of Exponential Functions :

$$\text{Recall: } \int e^{kx} dx = \frac{1}{k} e^{kx} + C$$

$$\text{In general: } \int a^x dx = ? \quad \text{for } a > 0$$

$$\text{Recall: } a^x = e^{\ln a^x} = e^{(\ln a)x}$$

$$\begin{aligned} \therefore \int a^x dx &= \int e^{(\ln a)x} dx \\ &= \frac{1}{\ln a} e^{(\ln a)x} + C \\ &= \frac{a^x}{\ln a} + C \end{aligned}$$

$$\text{OR: Recall that } \frac{d}{dx} a^x = a^x \ln a$$

$$\text{so } \frac{d}{dx} \frac{a^x}{\ln a} = a^x, \text{ and } \int a^x dx = \frac{a^x}{\ln a} + C$$

Integration of Logarithmic Functions :

$$\int \ln x dx = ? \quad \text{for } x > 0$$

$$\text{Ex: } \frac{d}{dx} x \ln x - x$$

$$\text{Ans: } \ln x !$$

$$\text{Therefore, } \int \ln x dx = x \ln x - x + C$$

Problem: How do we know $\frac{d}{dx} x \ln x - x = \ln x$ in advance?

(Make a guess of antiderivative of $\ln x$ directly)

Any direct way to find an antiderivative of $\ln x$? (Yes, later!)

e.g. (Constant issue)

$$\begin{array}{lll} \int (x+1)^2 dx & \text{let } u = x+1 & \int (x+1)^2 dx \\ = \int u^2 du & du = dx & = \int x^2 + 2x + 1 dx \\ = \frac{1}{3} u^3 + C & & = \frac{1}{3} x^3 + x^2 + x + C \\ = \frac{1}{3} (x+1)^3 + C & & \\ = \frac{1}{3} x^3 + x^2 + x + \frac{1}{3} + C & \swarrow \text{seems to} & \\ & \text{be different!} & \end{array}$$

Ans: This C is NOT that C !

Integration of Rational Functions :

• $\int \frac{p(x)}{ax+b} dx$

By long division, $p(x) = (ax+b)q(x) + R$

$$\frac{p(x)}{ax+b} = q(x) + \frac{R}{ax+b}$$

$$\begin{array}{r} q(x) \\ ax+b \overline{) p(x)} \\ \underline{} \\ R \end{array}$$

Then $\int \frac{p(x)}{ax+b} dx = \int q(x) + \frac{R}{ax+b} dx$

We know how to integrate!

e.g. $\int \frac{x^2+3x+5}{x+1} dx$

$$\begin{array}{r} x+2 \\ x+1 \overline{) x^2+3x+5} \\ \underline{x^2+x} \\ 2x+5 \\ \underline{2x+2} \\ 3 \end{array}$$

$$= \int x+2 + \frac{3}{x+1} dx$$

$$= \frac{x^2}{2} + 2x + 3 \ln|x+1| + C$$

$$\therefore x^2+3x+5 = (x+1)(x+2) + 3$$

$$\frac{x^2+3x+5}{x+1} = x+2 + \frac{3}{x+1}$$

Ex: Evaluate $\int \frac{6x^2-5x+1}{3x-2} dx$

Ans: $x^2 - \frac{x}{3} + \frac{1}{9} \ln|3x-2| + C$

• $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)} dx$

Express $\frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}$ into the form $\frac{A}{r_1x+s_1} + \frac{B}{r_2x+s_2}$.

Then $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)} dx = \int \frac{A}{r_1x+s_1} + \frac{B}{r_2x+s_2} dx$

We know how to integrate!

e.g. $\int \frac{5x-7}{x^2-2x-3} dx$

Note: $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$$\Rightarrow 5x-7 \equiv A(x+1) + B(x-3)$$

$$\Rightarrow A=3, B=2.$$

$$\int \frac{5x-7}{x^2-2x-3} dx = \int \frac{3}{x-3} + \frac{2}{x+1} dx = 3 \ln|x-3| + 2 \ln|x+1| + C$$

Ex: Evaluate $\int \frac{40}{x(200-x)} dx$

Ans: $\frac{1}{5} (\ln|x| - \ln|200-x|) + C = \frac{1}{5} \ln \left| \frac{x}{200-x} \right| + C$

• $\int \frac{ax+b}{(px+q)^2} dx$

Express $\frac{ax+b}{(px+q)^2}$ into the form $\frac{A}{(px+q)^2} + \frac{B}{px+q}$

Then $\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)} dx = \int \frac{A}{(px+q)^2} + \frac{B}{px+q} dx$
We know how to integrate!

e.g. $\int \frac{2x-1}{(x-2)^2} dx$

Suppose $\frac{2x-1}{(x-2)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2}$

$\Rightarrow 2x-1 = A+B(x-2)$

$\Rightarrow A=3, B=2$

$\int \frac{2x-1}{(x-2)^2} dx = \int \frac{3}{(x-2)^2} + \frac{2}{x-2} dx = \frac{-3}{x-2} + 2\ln|x-2| + C$

Ex: Evaluate $\int \frac{4x+2}{(2x-1)^2} dx$

Ans: $\frac{-2}{2x-1} + \ln|2x-1| + C$

Remarks:

If $\deg p(x) > 1$, $\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)} dx = ?$

Hint: Long division.

$\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)} dx = \int q(x) + \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)} dx$

reduced to previous case!

• $\int \frac{rx+s}{ax^2+bx+c} dx$ where $b^2-4ac < 0$

$$\int \frac{1}{x^2+a^2} dx \quad \text{let } x=au$$

$$= \int \frac{1}{a^2u^2+a^2} a du \quad dx = a du$$

$$= \frac{1}{a} \int \frac{1}{u^2+1} du$$

$$= \frac{1}{a} \tan^{-1} u + C$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{x^2+2x+5} dx$$

$$= \int \frac{1}{(x+1)^2+2^2} dx \quad \text{let } u=x+1$$

$$du = dx$$

$$= \int \frac{1}{u^2+2^2} du$$

$$= \frac{1}{2} \tan^{-1} \frac{u}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{x+1}{2} + C$$

$$\int \frac{4x+7}{x^2+2x+5} dx \quad \text{Note: } d(x^2+2x+5) = (2x+2) dx$$

$$\text{and } 4x+7 = 2(2x+2) + 3$$

$$= \int \frac{2(2x+2) + 3}{x^2+2x+5} dx$$

$$= 2 \int \frac{2x+2}{x^2+2x+5} dx + 3 \int \frac{1}{x^2+2x+5} dx$$

$$= 2 \ln(x^2+2x+5) + 3 \left(\frac{1}{2} \tan^{-1} \frac{x+1}{2} \right) + C$$

$$= 2 \ln(x^2+2x+5) + \frac{3}{2} \tan^{-1} \frac{x+1}{2} + C$$

Remarks:

If $\deg p(x) > 1$, $\int \frac{p(x)}{ax^2+bx+c} dx = ?$ where $b^2-4ac < 0$

Hint: Long division.

Integration of Trigonometric Functions:

• $\int \tan x \, dx$ and $\int \cot x \, dx$

$$\begin{aligned} & \int \tan x \, dx \\ &= \int \frac{\sin x}{\cos x} \, dx \quad \text{let } u = \cos x \\ &= \int -\frac{1}{u} \, du \quad \frac{du}{dx} = -\sin x \\ &= -\ln|u| + C \quad -du = \sin x \, dx \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C \end{aligned}$$

Ex: $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} \, dx \quad \text{let } u = \sin x$$

Ex: :

$$= \ln|\sin x| + C$$

• $\int \sec x \, dx$ and $\int \csc x \, dx$, t-formula

t-formula:

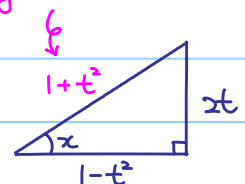
$$\text{Let } t = \tan \frac{x}{2}$$

Idea: We can express all trigonometric functions in terms of t.

$$\text{Note: } \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{2t}{1-t^2} \quad \text{and so } \cot x = \frac{1-t^2}{2t}$$

$$\begin{aligned} \therefore \sin x &= \frac{2t}{1+t^2} & \text{and so} & \quad \csc x = \frac{1+t^2}{2t} \\ \cos x &= \frac{1-t^2}{1+t^2} & & \quad \sec x = \frac{1+t^2}{1-t^2} \end{aligned}$$

By Pyth. thm.



Therefore, all trigonometric functions in terms of t.

Note: $t = \tan \frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2} = \frac{1}{2}(1+t^2)$$

$$dx = \frac{2}{1+t^2} dt$$

Idea: $\int f(\sin x, \cos x) \, dx$

$$= \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2} dt$$

Rational functions of t.

Transforming an integral of trigonometric function to an integral of rational function.

$$\begin{aligned}
& \int \csc x \, dx \\
&= \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{1}{t} dt \\
&= \ln|t| + C \\
&= \ln\left|\tan \frac{x}{2}\right| + C
\end{aligned}$$

$$\begin{aligned}
& \int \sec x \, dx \\
&= \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2}{1-t^2} dt \\
&= \int \frac{1}{1+t} + \frac{1}{1-t} dt \\
&= \ln|1+t| - \ln|1-t| + C \\
&= \ln\left|\frac{1+t}{1-t}\right| + C \\
&= \ln\left|\frac{2t+1-t^2}{1-t^2}\right| + C \\
&= \ln|\tan x + \sec x| + C
\end{aligned}$$

e.g. $\int \frac{1}{1+\cos x} dx$

$$\begin{aligned}
& \int \frac{1}{1+\cos x} dx \\
&= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\
&= \int dt \\
&= t + C \\
&= \tan \frac{x}{2} + C
\end{aligned}$$