Talyor Polynomials

Let $f(x)$ be a function with derivatives of all orders on an open interval I, and $a \in I$.

Goal: Can we approximate $f(x)$ around the point a by a polynomial $P_n(x)$ of degree n

in a sense that
$$\left.\begin{aligned} f(a) &= P_n(a) \\ f'(a) &= P_n'(a) \\ &\vdots \\ f^{(n)}(a) &= P_n^{(n)}(a) \end{aligned}\right\} n+1 \text{ conditions}$$
?

Let $P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

$$= \sum_{i=1}^{n} a_i (x-a)^i$$

$\underbrace{a_0, a_1, \cdots, a_n}_{n+1 \text{ constants.}}$ are constants to be detemined.

Remark: $n+1$ conditions, $n+1$ constants $\Rightarrow$ a_i's are completely determined.

To determine a_i's :

- $P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

  $f(a) = P_n(a) = a_0$

- $P_n'(x) = 1\,a_1 + 2\,a_2(x-a) + 3\,a_3(x-a)^2 + \cdots + n\,a_n(x-a)^{n-1}$

  $f'(a) = P_n'(a) = 1\,a_1$

  $$a_1 = \frac{f'(a)}{1}$$

- $P_n''(x) = 2\cdot 1\,a_2 + 3\cdot 2\,a_3\,(x-a) + \cdots + n\cdot(n-1)\,a_n(x-a)^{n-2}$

  $f'(a) = P_n'(a) = 2!\,a_2$

  $$a_2 = \frac{f''(a)}{2!}$$

Repeating the process, in general, we have $a_k = \frac{f^{(k)}(a)}{k!}$ $\quad k = 0, 1, 2, \cdots, n$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the Talyor polynomial of order n generated by f at a.

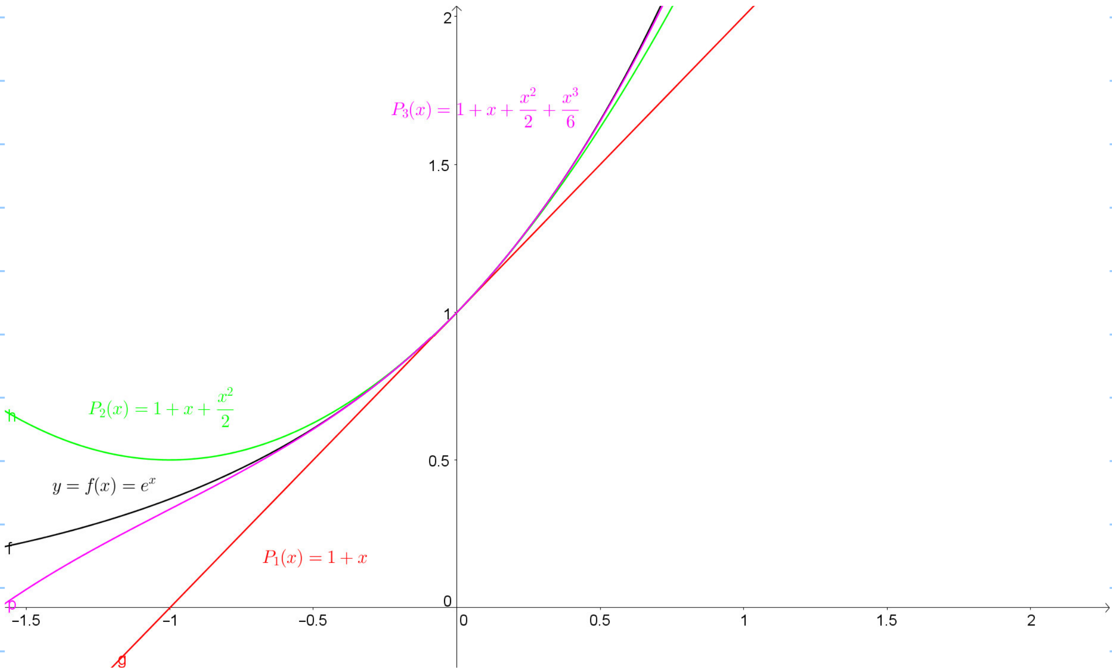
e.g. Let $f(x) = e^x$, find the Talyor polynomials $P_n(x)$ generated by f at $x=0$.

Note: $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$ for $k = 0, 1, 2, \cdots, n$

$$\therefore \quad P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{1}{k!}x^k$$



$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
$P_2(x) = 1 + x + \frac{x^2}{2}$
$y = f(x) = e^x$
$P_1(x) = 1 + x$
h
f
p
g
2
1.5
1
0.5
0
−1.5
−1
−0.5
0
0.5
1
1.5
2

e.g. Let $f(x) = \cos x$, find the Talyor polynomials generated by f at $x=0$.

Note : $f(x) = \cos x$ $\qquad f'(x) = -\sin x$

$f''(x) = -\cos x$ $\qquad f'''(x) = \sin x$

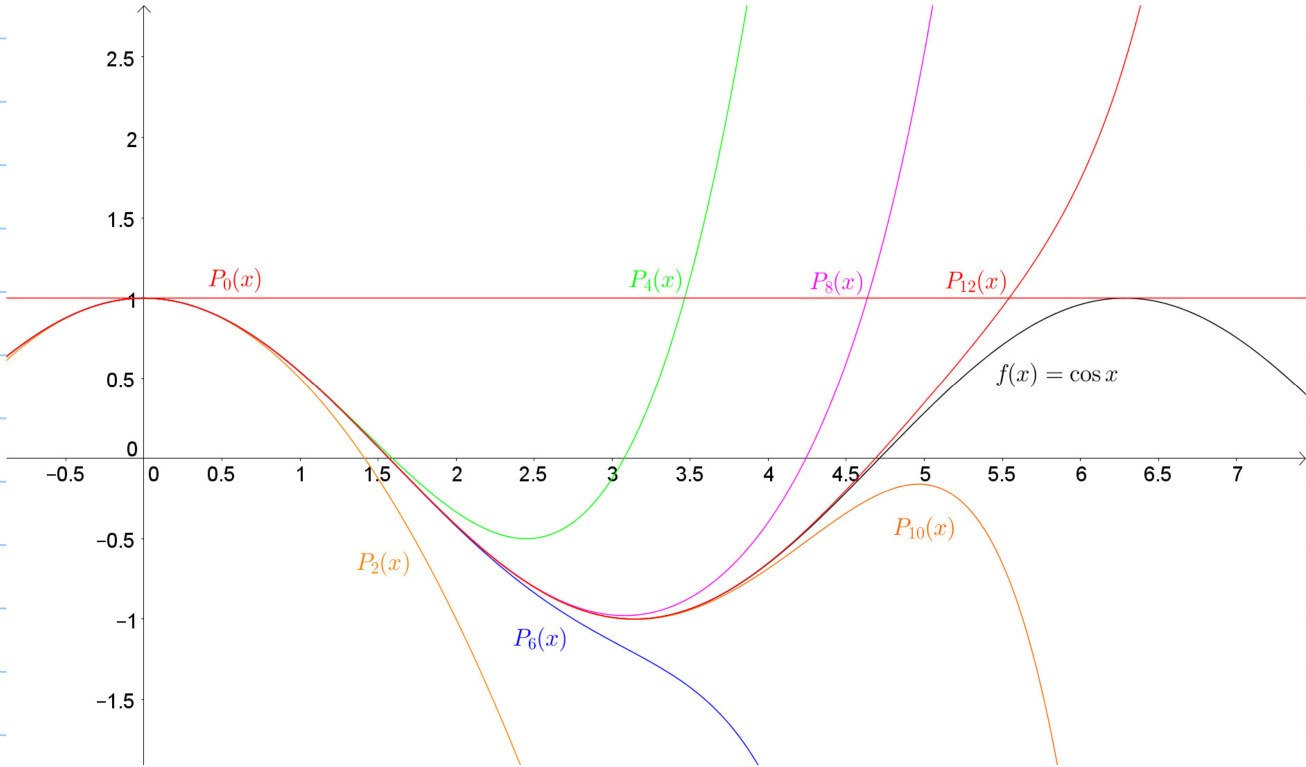
$f^{(4)}(x) = \cos x$ $\qquad f^{(5)}(x) = -\sin x$

$\vdots \qquad \vdots$

$f^{(2n)}(x) = (-1)^n \cos x$ $\qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

$\therefore\ f^{(2n)}(0) = (-1)^n$ $\qquad f^{(2n+1)}(0) = 0$

$$\therefore\ P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



$P_0(x)$
$P_2(x)$
$P_4(x)$
$P_6(x)$
$P_8(x)$
$P_{10}(x)$
$P_{12}(x)$
$f(x) = \cos x$

Ex: Find the Talyor polynomials generated by f at $x=0$, if $f(x)=$

a) $\sin x$

b) $\frac{1}{1-x}$

c) $\ln(1+x)$

When we approximate $f(x)$ by $P_n(x)$, there is an error term $E_n(x) = f(x) - P_n(x)$

The error term tells us how good / bad our approximation is!

The error term can be described by the following theorem.

Talyor's Theorem

Theorem (Talyor's Theorem):

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b, $f^{(n)}$ is differentiable on the open interval between a and b, then there exists c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

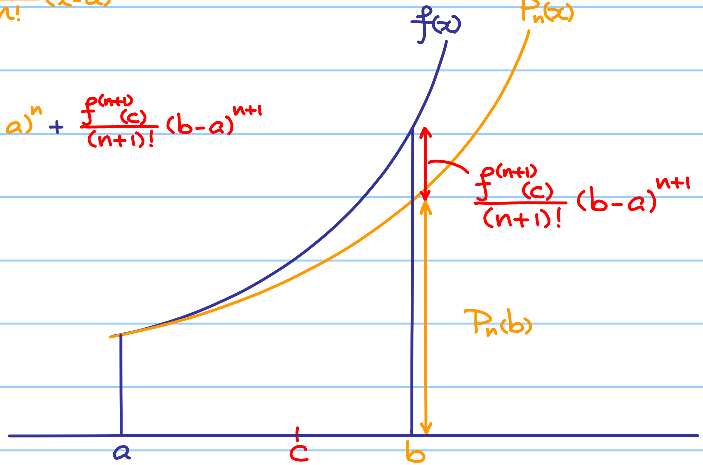
Remark: If $n=0$,

$$f(b) = f(a) + f'(c)(b-a)$$

it is just Mean Value Theorem.

Let $P_n(x)$ be the Talyor polynomial of order n generated by f at the point a.

i.e. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$



$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

$$f(b) = P_n(b) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}_{\text{error}}$$

Replace b by x, and let x varies

$$f(x) = P_n(x) + \underbrace{\frac{f^{(n+1)}(c(x))}{(n+1)!}(x-a)^{n+1}}_{\text{error } E_n(x)}$$

$c(x)$ lies between a and x.

proof :

Assume $b > a$.

Let $P_n(x)$ be the Talyor polynomial of order n generated by f at the point a.

i.e. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Let $F(x) = f(x) - P_n(x) - \frac{f(b)-P_n(b)}{(b-a)^{n+1}}(x-a)^{n+1}$

Check : F is continuous on $[a,b]$

F is differentiable on (a,b)

$F(a) = F(b) = 0$

Apply Rolle's Theorem, $\exists\ c_1 \in (a,b)$ such that $F'(c_1) = 0$

Check : F' is continuous on $[a,b]$

F' is differentiable on (a,b)

$F'(a) = F'(c_1) = 0$

Apply Rolle's Theorem , $\exists\, c_2 \in (a, c_1)$ such that $F''(c_2) = 0$

Repeating the process : $\exists\, c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c_{n+1}) = 0$

Note : $F^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\,\dfrac{f(b)-P_n(b)}{(b-a)^{n+1}}$

$$0 = F^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1}) - (n+1)!\,\frac{f(b)-P_n(b)}{(b-a)^{n+1}}$$

$$\therefore\ \frac{f^{(n+1)}(c_{n+1})}{(n+1)!} = \frac{f(b)-P_n(b)}{(b-a)^{n+1}}$$

$$\therefore \quad F(x) = f(x) - P_n(x) - \frac{f(b) - P_n(b)}{(b-a)^{n+1}}(x-a)^{n+1}$$

$$F(x) = f(x) - P_n(x) - \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(x-a)^{n+1}$$

Recall $F(b) = 0$, so $f(b) = P_n(b) + \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$$

The proof for the case $a > b$ is similar.

e.g. Approximate $\cos 0.1$

Let $f(x) = \cos x$,

$P_5(x) = P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ Talyor polynomials generated by f at $x=0$.

$\cos 0.1 = f(0.1) \approx P_5(0.1) = 0.995004166\cdots$

By Talyor's Theorem $f(0.1) = P_5(0.1) + \frac{f^{(6)}(c)}{6!}(0.1)^6$ $c \in (0, 0.1)$

Absolute Error $= \left| \frac{f^{(6)}(c)}{6!}(0.1)^6 \right|$

$\leq \frac{1}{6!}(0.1)^6 \approx 1.38 \times 10^{-9}$

Very small.

Note : $f^{(6)}(x) = -\cos x$

$\Rightarrow |f^{(6)}(c)| \leq 1$

Question: If we want to approximate $\cos 2$ by $P_n(2)$ with the same precision, i.e. absolute error $\leq 1.38 \times 10^{-9}$, then what is the least n?

Solve n

$$\text{Absolute Error} = \left| \frac{f^{(n+1)}(c)}{(n+1)!} 2^{n+1} \right| \leq \frac{2^{n+1}}{(n+1)!} \leq 1.38 \times 10^{-9} \qquad c \in (0,2)$$

$n = 16$! Much more terms needed!

e.g. Let $f(x) = \frac{1}{1-x}$

Ex: Suppose $P_n(x)$ is Talyor polynomial of order n generated by f at 0.

Show that $P_n(x) = 1 + x + x^2 + \cdots + x^n$

Note: $f(0.1) = \frac{1}{1-0.1} = 1.111\cdots$

$P_n(0.1) = 1 + 0.1 + 0.1^2 + \cdots + 0.1^n = 1.\underbrace{111\cdots1}_{n}$

$E_n(0.1) = f(0.1) - P_n(0.1)$ is getting closer and closer to 0 as n increases.

Good Approximation

$f(2) = \frac{1}{1-2} = -\frac{1}{2}$

$P_n(2) = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

$E_n(2) = f(2) - P_n(2)$ is NOT getting closer and closer to 0 as n increases.

Bad Approximation

We express $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + E_n(x)$

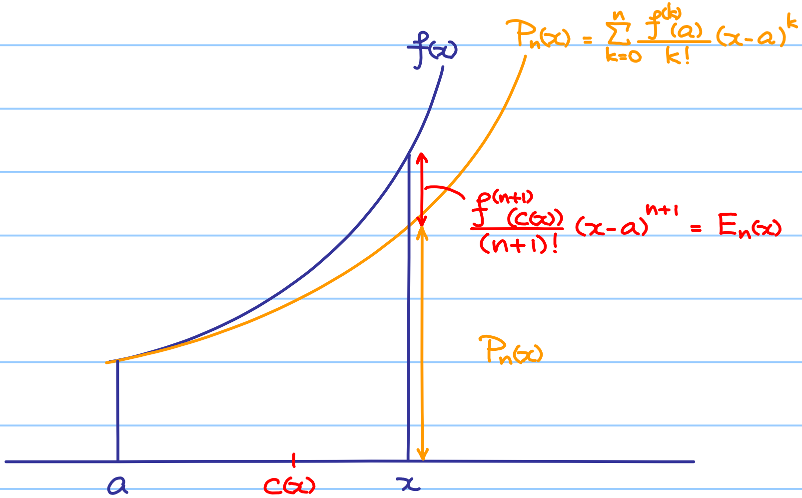
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x)$$

Fix x, then $E_n(x)$ becomes a sequence of real numbers.

If $\lim_{n\to\infty} E_n(x) = 0$, that means the error is getting closer and closer to 0 as we increase the number of terms to approximate $f(x)$.

If $\lim_{n\to\infty} E_n(x) = 0$ for all $x \in I$, we have $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

If $\lim_{n\to\infty} E_n(x) = 0$ for all $x \in I$, we have $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$



$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is said to be the Taylor series generated by f at a.

We say the Talyor series converges to $f(x)$ for all $x \in I$ if $\lim_{n\to\infty} E_n(x) = 0$

e.g. Let $f(x) = \cos x$

$$f(x) = \cos x = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}}_{P_{2n}(x) = P_{2n+1}(x)} + E_{2n+1}(x)$$

$$0 \leq |E_{2n+1}(x)| = \left| \frac{f^{(2n+2)}(c(x))}{(2n+2)!} x^{2n+2} \right| \leq \frac{|x^{2n+2}|}{(2n+2)!}$$

Note : $|f^{(2n+2)}(c(x))| = |\cos(c(x))| \leq 1$

$$\lim_{n\to\infty} \frac{|x^{2n+2}|}{(2n+2)!} = 0 \quad \text{— } (*)$$

By sandwich theorem, $\lim_{n\to\infty} |E_{2n+1}(x)| = 0$ and hence $\lim_{n\to\infty} E_{2n+1}(x) = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k} \quad \text{for all } x \in \mathbb{R}$$

(*) For a fix x, there exists $K \in \mathbb{N}$ such that $|x| < K$

$$0 \leq \frac{|x^{2n+2}|}{(2n+2)!} = \underbrace{\frac{|x^K|}{K!}}_{M} \cdot \frac{|x|}{K+1} \cdot \frac{|x|}{K+2} \cdot \dots \cdot \frac{|x|}{2n+2} \leq M \left(\frac{|x|}{K+1}\right)^{2n+2-K}$$

Note: As $0 < \frac{|x|}{K+1} < 1$, $\lim_{n\to\infty} M \left(\frac{|x|}{K+1}\right)^{2n+2-K} = 0$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty} \frac{|x^{2n+2}|}{(2n+2)!} = 0$

Remark: In general, let $\alpha \in \mathbb{R}$, we have $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

Frequently used Talyor series :

1) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \,, \qquad |x| < 1$

2) $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n \,, \qquad |x| < 1$

3) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,, \qquad \forall x \in \mathbb{R}$

4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \,, \qquad \forall x \in \mathbb{R}$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n} \,, \qquad \forall x \in \mathbb{R}$

6) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{1}{n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n} \,, \qquad \forall\ -1 < x \leq 1$

e.g. (NOT Rigorous)

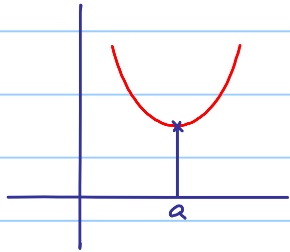
Suppose $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

in an interval I and $a \in I$.

If we know $f'(a) = 0$ and $f''(a) > 0$,

then if $x \sim a$,

$$f(x) \approx f(a) + \cancel{f'(a)}^{\,0}(x-a) + \underbrace{\frac{f''(a)}{2!}}_{>0}(x-a)^2$$



locally, like a parabola opening upward!

It suggests why $f(x)$ attains minimum at $x = a$.

How about $f''(a) = 0$?

e.g. (NOT Rigorous)

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}$$

$$= 1 \underbrace{- \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots}_{\text{terms involves } x}$$

It suggests $\lim_{x \to 0} \frac{\sin x}{x} = 1$

In general, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

Suppose $f(a) = g(a) = 0$ and $f'(a), g'(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots}$$

$$= \lim_{x\to a} \frac{f'(a) + \text{terms involves } (x-a)}{g'(a) + \text{terms involves } (x-a)} = \frac{f'(a)}{g'(a)}$$

The formal statement : L'hôpital Rule

Intermediate Form and L'hôpital Rule

Intermediate Form $\frac{0}{0}$ and L'hôpital Rule

Consider $\lim_{x \to a} \frac{f(x)}{g(x)}$ and suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Case 1: If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

Case 2: If $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does NOT exist.

Case 3: If $\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = 0$, then we do NOT know whether $\lim_{x \to a} \frac{f(x)}{g(x)}$ exist!

We call it interminate form $\frac{0}{0}$.

Theorem (L'hôpital's Rule)

Suppose that $f(a) = g(a) = 0$, I is an open interval containing a, f and g are differentiable on $I \setminus \{a\}$, and $g'(x) \neq 0$ on $I \setminus \{a\}$.

If $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(Further, if $f'(x)$ and $g'(x)$ are continuous at a and $g'(a) \neq 0$,

then $\lim\limits_{x\to a} f'(x) = f'(a)$ and $\lim\limits_{x\to a} g'(x) = g'(a)$

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \quad)$$

e.g. $\lim_{x\to 0} \frac{\sin x}{x}$ $\left(\frac{0}{0}\right)$ — (*)

$= \lim_{x\to 0} \frac{\cos x}{1}$ — (**)

$= \frac{1}{1}$

$= 1$

Logic: limit (**) exists $\Rightarrow$ limit (*) exists

e.g. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\sin x}{2x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\cos x}{2}$

$= \frac{1}{2}$

Intermediate Form $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$

- L'hôpital's Rule can also be applied to $\frac{\infty}{\infty}$
- L'hôpital's Rule can also be applied to left hand limit or right hand limit

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \quad , \quad \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}$$

- L'hôpital's Rule can also be applied to limits at infinities

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to +\infty} \frac{f'(x)}{g'(x)} \quad , \quad \lim_{x\to -\infty} \frac{f(x)}{g(x)} = \lim_{x\to -\infty} \frac{f'(x)}{g'(x)}$$

e.g. Intermediate Form $\frac{\infty}{\infty}$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{1 + \tan x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \sin x$$

$$= 1$$

$$\lim_{x \to +\infty} \frac{\ln x}{2\sqrt{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to +\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x \to +\infty} \frac{1}{\sqrt{x}}$$

$$= 0$$

e.g. Intermediate Form $\infty \cdot 0$

Idea: Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to +\infty} x \sin \frac{1}{x} \quad (\infty \cdot 0)$$

$\downarrow$ convert to

$$= \lim_{x \to +\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to +\infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to +\infty} \cos \frac{1}{x}$$

$$= 1$$

Alternative method :

$$\lim_{x \to +\infty} x \sin \frac{1}{x} \quad (\infty \cdot 0)$$

$$= \lim_{h \to 0^+} \frac{\sin h}{h} \quad \left(\frac{0}{0}\right)$$

Let $h = \frac{1}{x}$,

As $x \to +\infty$, $h \to 0^+$

$$= 1$$

$$\lim_{x \to 0^+} \sqrt{x} \ln x \qquad (\infty \cdot 0)$$

$\downarrow$ convert to

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}}$$

$$= \lim_{x \to 0^+} -2\sqrt{x}$$

$$= 0$$

Remark: Why don't we try $\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\sqrt{x}}{\left(\frac{1}{\ln x}\right)}$ $\left(\frac{0}{0}\right)$?

e.g. Intermediate Form $\infty - \infty$

Idea : Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) \qquad (\infty - \infty)$$

$\downarrow$ convert to

$$= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \left(\frac{0}{0}\right)$$

Ex : $\vdots$

$= 0$

Intermediate Form 1^{∞}, 0^{0}, ∞^{0}

Idea: Taking ln, converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

e.g. Intermediate Form 1^{∞}

Find $\lim_{x\to 1^+} x^{\frac{1}{1-x}}$ $\quad (1^{\infty})$

Let $y = \ln x^{\frac{1}{1-x}}$

$= \frac{\ln x}{1-x}$

$\lim_{x\to 1^+} y = \lim_{x\to 1^+} \frac{\ln x}{1-x} \quad (\frac{0}{0})$

$= \lim_{x\to 1^+} \frac{(\frac{1}{x})}{-1}$

$= -1$

$\therefore \lim_{x\to 1^+} x^{\frac{1}{1-x}} = \lim_{x\to 1^+} e^{y} = e^{-1}$

e.g. Intermediate Form ∞^0

Find $\lim_{x \to +\infty} x^{\frac{1}{x}}$ $\quad (\infty^0)$

Let $y = \ln x^{\frac{1}{x}}$

$= \frac{\ln x}{x}$

$\lim_{x \to +\infty} y = \lim_{x \to +\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{x \to +\infty} \frac{\left(\frac{1}{x}\right)}{1}$

$= 0$

$\therefore \lim_{x \to +\infty} x^{\frac{1}{x}} = \lim_{x \to +\infty} e^{y} = e^{0} = 1$