e.g. Prove that $e^x \geq 1+x \quad \forall x \in \mathbb{R}$.

(i.e. $e^x - x - 1 \geq 0$)

Let $f(x) = e^x - x - 1$

(Want to find the global minimum of $f(x)$ and see if it is ≥ 0.)

$f'(x) = e^x - 1$

$f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$

f is strictly increasing when $x > 0$ and strictly decreasing when $x < 0$

(and f is continuous at $x = 0$.)

f attains minimum when $x = 0$ (By 1st derivative check)

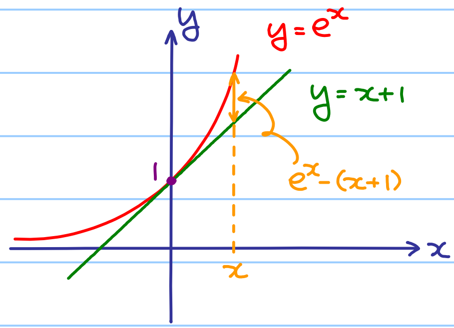
(In fact, global minimum, why?)

$\therefore f(x) \geq f(0) \quad \forall x \in \mathbb{R}$ —— (*)

$= e^0 - 0 - 1$

$= 0$

Note: The equality holds iff $x = 0$



Ex: By considering $f(x) = \sin x - x\cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

show that $\cos x < \frac{\sin x}{x}$ for $-\frac{\pi}{2} < x < 0$ or $0 < x < \frac{\pi}{2}$.

Stationary Points :

If $f'(a) = 0$, then $(a, f(a))$ is called a stationary point.

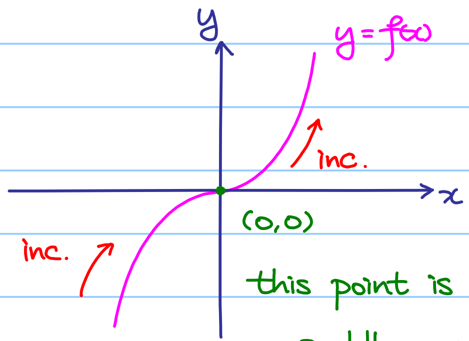
But even $f'(a) = 0$, it's still hard to say !

e.g. If $f(x) = x^3$, then $f'(x) = 3x^2$.

Note : 1) $f'(0) = 0$

2) $f'(x) = 3x^2 > 0$ for $x \neq 0$

i.e. No change of sign of $f'(x)$ at $x = 0$.



this point is called a saddle point

Note : a stationary is NOT necessary to be a max. / min. point !

Theorem : Let $f : (a,b) \longrightarrow \mathbb{R}$ be a function and $c \in (a,b)$ such that

1) $f'(c)$ exists

2) f attains maximum (or minimum) at $x = c$.

Then, we have $f'(c) = 0$.

proof : Assume f attains maximum at $x = c$.

$$f'(c) \text{ exist } \Rightarrow \lim_{\Delta x \to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c)$$

Note : $\frac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0$ for all $\Delta x > 0 \Rightarrow f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0$

$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0$ for all $\Delta x < 0 \Rightarrow f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0$

$\therefore f'(c) = 0$.

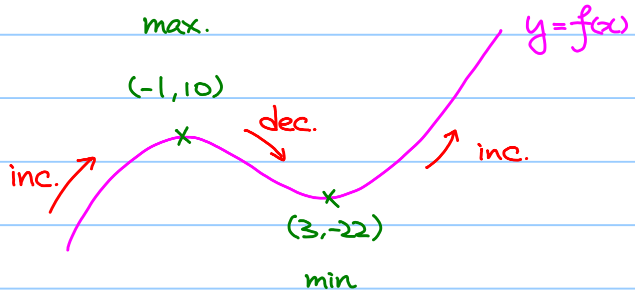
Remark : If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, solving the equation $f'(x) = 0$ is sufficient to capture all maximum and minimum.

e.g. If $f(x) = x^3 - 3x^2 - 9x + 5$

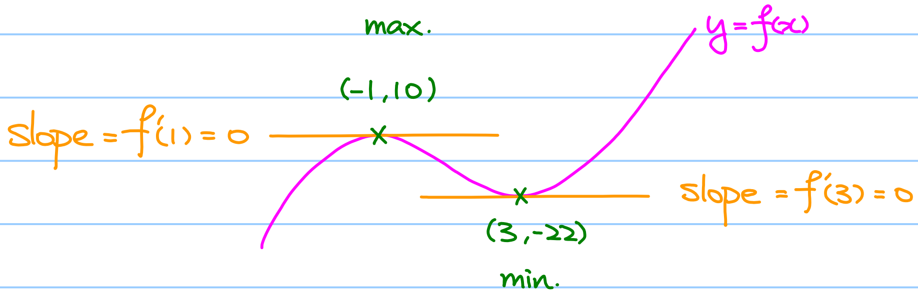
then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$



Furthermore,



Higher Derivatives :

$s(t)$: distance function (depends on time t)

(instantaneous) speed = rate of change of distance travelled with respect to t.

$$v(t) = \frac{ds}{dt}$$ (still a function of t)

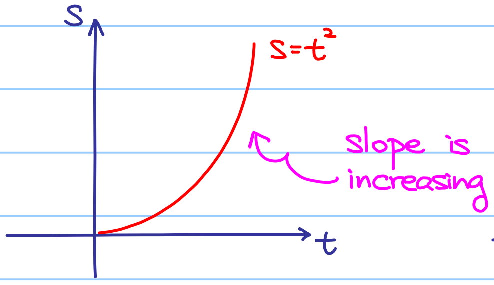
Question : What is $\frac{dv}{dt}$?

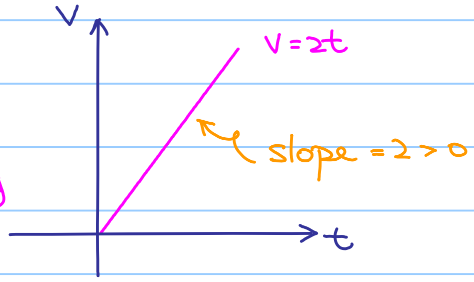
Answer : Acceleration !

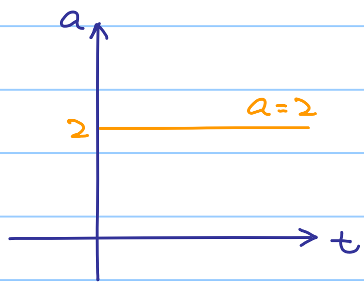
= rate of change of speed with respect to t.

We write $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

e.g. $s(t) = t^2$ $\qquad v(t) = \frac{ds}{dt} = 2t$ $\qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$







speed is increasing

i.e. accelerating

In general, let $y = f(x)$

We have : (1st derivative) $\qquad \frac{dy}{dx} = \frac{df}{dx} = f'(x)$

(2nd derivative) $\qquad \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

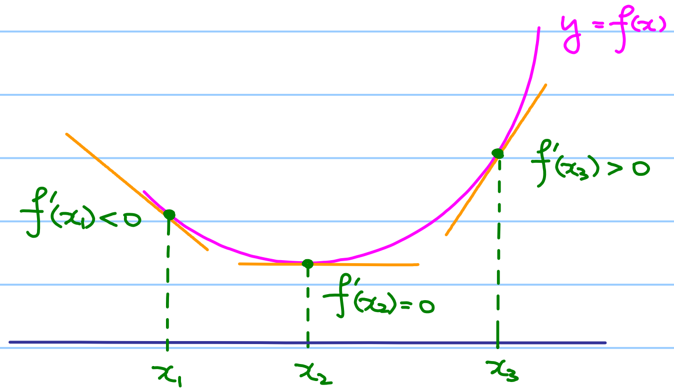
(nth derivative) $\qquad \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

2nd Derivative and Concavity :

Think : If $f''(x) > 0$ for $a<x<b$

then $f'(x)$ is strictly increasing on (a,b)

Picture :



Slope of the tangent line at $(x, f(x))$ increases as x increases !

(NOT $f(x)$ is increasing !)

If $f''(x) > 0$ for $a<x<b$,

then $f(x)$ is a concave function on (a,b).

Similarly : If $f''(x) < 0$ for $a<x<b$,

then $f(x)$ is a convex function on (a,b).

2nd Derivative Check :

Suppose $f(x)$ is twice differentiable at $x=a$. (i.e. $f'(a)$ and $f''(a)$ exist)

If (1) $f'(a)=0$ (i.e. $(a, f(a))$ is a stationary point.)

(2) $f''(a) < 0$ (Roughly speaking : $f(x)$ is convex near $x=a$,

if $f''(x)$ is continuous at $x=a$.)

then $(a, f(a))$ is a relative maximum.

We have similar result for relative minimum.

Caution : If $f''(a)=0$, then NO conclusion !

Consider $f(x) = x^4, x^3, -x^4$

We have $f'(0) = f''(0) = 0$ in each case, but $(0,0)$ is

- min. for the 1st case.
- saddle point for the 2nd case.
- max. for the 3rd case.

e.g. If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x-1)$

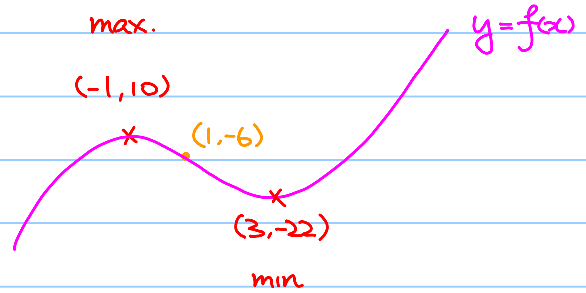
$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

$f''(x) = 6x - 6$

$f''(x) > 0$ if $x > 1$ $\qquad f''(-1) = 12 < 0$

$f''(x) < 0$ if $x < 1$ $\qquad f''(3) = 12 > 0$



		-1		3	
$f'(x)$	+ve		-ve		+ve
$f(x)$	inc.		dec.		inc.

		1	
$f''(x)$	-ve		+ve
$f(x)$	convex		concave

Note : The curve changes from being convex to concave at $(1,6)$.

This point is called a point of inflection.

Point of inflection :

Suppose $f(x)$ is continuous at $x = a$ and differentiable on some open interval I containing $x = a$, except possibly at $x = a$ itself.

If $f''(x) > 0$ (resp. $f''(x) < 0$) for all x in I with $x < a$, and

$f''(x) < 0$ (resp. $f''(x) > 0$) for all x in I with $x < a$,

then $(a, f(a))$ is a point of inflection.

(Remember the slogan : Change sign of $f''(x)$ at $x = a$.)

e.g. $f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$

Find the range of x such that

(1) $f'(x) > 0$, $f'(x) < 0$

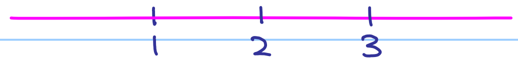
(2) $f''(x) > 0$, $f''(x) < 0$

Step 1 : Find $f'(x)$ and factorize it.

$$f'(x) = 60x^4 - 420x^3 + 1020x^2 - 1020x + 360$$
$$= 60(x^4 - 7x^3 + 17x^2 - 17x + 6)$$
$$= 60(x-1)^2(x-2)(x-3)$$

(Using factor theorem)

Step 2 :



$\downarrow$ gives intervals

$x<1$ $1<x<2$ $2<x<3$ $x>3$

(Reason : those factors may change sign at the boundaries of the intervals.)

Step 3 :

	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$(x-1)^2$	+	0	+	+	+	+	+
$(x-2)$	−	−	−	0	+	+	+
$(x-3)$	−	−	−	−	−	0	+
$f'(x)$	+	0	+	0	−	0	+
$f(x)$	inc	saddle pt.	inc.	max.	dec.	min	inc.

saddle point = $(1, -23)$

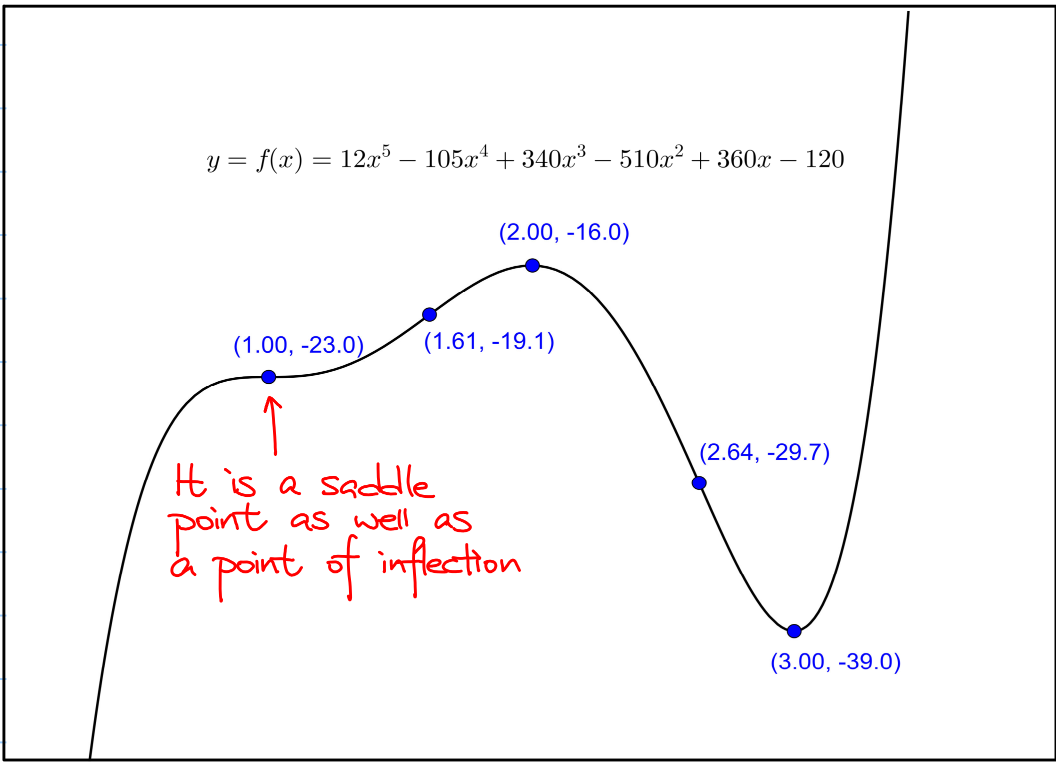
max = $(2, -16)$ min = $(3, -39)$

Similarly,

$$f''(x) = 240x^3 - 1260x^2 + 2040x - 1020$$
$$= 60(x-1)(4x^2 - 17x + 17)$$
$$= 240(x-1)\left[x - \left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x - \left(\frac{17-\sqrt{17}}{8}\right)\right]$$

$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
x		1		$\frac{17-\sqrt{17}}{8} \approx 1.61$		$\frac{17+\sqrt{17}}{8} \approx 2.64$	
$f(x)$	$\frown$		$\smile$		$\frown$		$\smile$

points of inflection : $(1, -23)$, $\left(\frac{17 \pm \sqrt{17}}{8}, f\left(\frac{17 \pm \sqrt{17}}{8}\right)\right)$



e.g. $\quad f(x) = \dfrac{x}{(x+1)^2} \qquad x \neq -1$

$$f'(x) = \frac{1-x}{(x+1)^3}$$

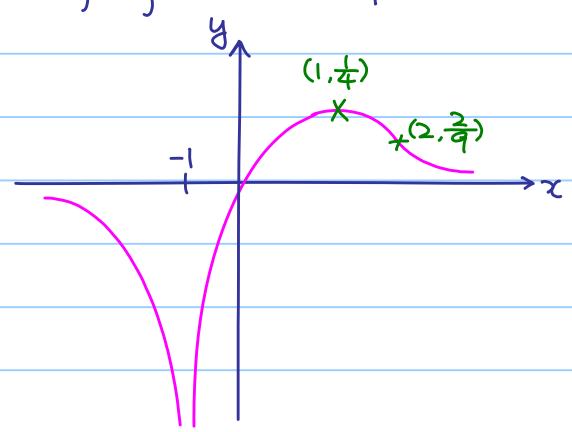
		-1		1	
$f'(x)$	$-$	NOT defined	$+$	0	$-$
$\Downarrow$					
$f(x)$	dec.	NOT defined	inc.	max.	dec.

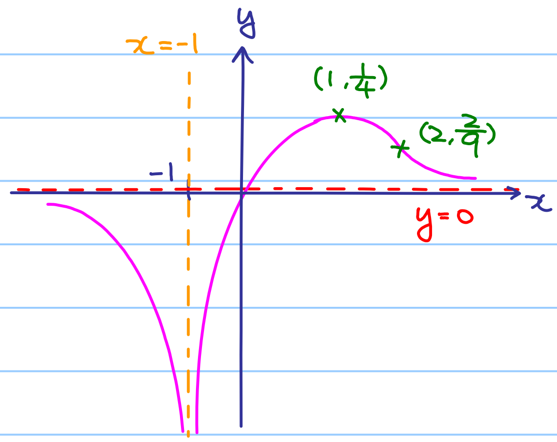
max. $= \left(1, \frac{1}{4}\right)$

$$f''(x) = \frac{2(x-2)}{(x+1)^4}$$

		-1		2	
$f''(x)$	$-$	NOT defined	$-$	0	$+$
$\Downarrow$					
$f(x)$	$\cap$		$\cap$		$\cup$

point of inflection : $\left(2, \frac{2}{9}\right)$





Note : The graph of $y=f(x)$ behaves like

- the vertical line $x=-1$, when x is "near" -1.
- the horizontal line $y=0$, when x is "near $+\infty$ or $-\infty$".

In fact, $x=-1$ is called a vertical asymptote,

$y=0$ is called a horizontal asymptote.

Finding vertical asymptote :

If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$, then $x=a$ is called a vertical asymptote.

Finding horizontal asymptote :

If $\lim_{x\to+\infty} f(x) = L$, where L is a real number, then $y=L$ is a horizontal asymptote.

(Similar for $\lim_{x\to-\infty} f(x)$)

Note : It may happen that both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist

but they are NOT the same.