

e.g. If  $f(x)$  is differentiable, then  $k \cdot f(x)$  is differentiable.  
and  $(kf)'(x) = k \cdot f'(x)$  (or write  $\frac{d}{dx} k \cdot f(x) = k \frac{df}{dx}$ ).



Idea: Let  $g(x) = k$ , then  $g'(x) = 0$ .

Apply product rule, the result follows.

e.g. Find  $\frac{d}{dx} (3x^2 + 7x - 2)$

$$\frac{d}{dx} (3x^2 + 7x - 2) = \frac{d}{dx} (3x^2) + \frac{d}{dx} (7x) - \frac{d}{dx} (2)$$

Apply ① and ②

$$= 3 \frac{d}{dx} (x^2) + 7 \frac{d}{dx} (x) - \frac{d}{dx} (2)$$

$$= 3(2x) + 7(1) - 0$$

$$= 6x + 7$$

e.g. Find the derivative of the function  $(3x^2-5x+1)(2x+7)$

$$\frac{d}{dx} [(3x^2-5x+1)(2x+7)]$$

$$= \left[ \frac{d}{dx} (3x^2-5x+1) \right] (2x+7) + (3x^2-5x+1) \left[ \frac{d}{dx} (2x+7) \right]$$

Apply ③ product rule

$$= (6x-5)(2x+7) + (3x^2-5x+1)(2)$$

$$= 18x^2 + 22x - 33$$

Ex: Try to compare : Expand  $(3x^2-5x+1)(2x+7)$  and get  $6x^3+11x^2-33x+7$

Then differentiate , get the same result ?

e.g. Find the derivative of the function  $\frac{2x}{x^2+1}$ .

$$\frac{d}{dx} \frac{2x}{x^2+1} = \frac{\left[\frac{d}{dx}(2x)\right](x^2+1) - (2x)\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

eg. Find  $\frac{d}{dx}(\frac{1}{\sqrt{x}} + \sqrt{x})$

$$\frac{d}{dx}(\frac{1}{\sqrt{x}} + \sqrt{x}) = \frac{d}{dx}(x^{-\frac{1}{2}} + x^{\frac{1}{2}})$$

$$= -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

## Derivatives of Trigonometric Function :

Preparations :

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{1}{2} \frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2} \\ &= \frac{1}{2}\end{aligned}$$

Note:  $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$\therefore 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot x \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \lim_{x \rightarrow 0} x \\ &= \frac{1}{2} \cdot 0 \\ &= 0\end{aligned}$$

Let  $f(x) = \cos x$

$$\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \cos x \frac{\cos \Delta x - 1}{\Delta x} - \sin x \frac{\sin \Delta x}{\Delta x}$$

$$= -\sin x \quad (\because \lim_{\Delta x \rightarrow 0} \frac{\cos \Delta x - 1}{\Delta x} = 0 \text{ and } \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} = 1.)$$

$$\therefore \frac{d}{dx} \cos x = -\sin x$$

Ex: Show  $\frac{d}{dx} \sin x = \cos x$  by using method similar to the above.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x} \quad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x}$$

:

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Ex: By quotient rule.

$$\text{Ex: } \frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

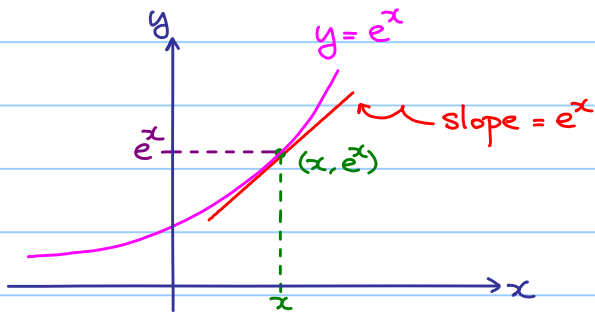
$$\frac{d}{dx} \cot x = -\csc^2 x$$

Derivative of  $e^x$  :

Recall :  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Cheating :  $\frac{d}{dx} e^x = \frac{d}{dx} (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)$   
 $= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$   
 $= e^x$  (getting back itself)

Geometrical meaning :





e.g. Find  $\frac{d}{dx} [e^x(3x^2+7x-2)]$

$$\begin{aligned}\frac{d}{dx} [e^x(3x^2+7x-2)] &= \left[ \frac{d}{dx} e^x \right] (3x^2+7x-2) + e^x \left[ \frac{d}{dx} (3x^2+7x-2) \right] \\ &= e^x(3x^2+7x-2) + e^x(6x+7) \\ &= e^x(3x^2+13x+5)\end{aligned}$$

Question: How to differentiate a more complicated function, such as  $\sqrt{x^2+3x}$  ?

We need a tool called **chain rule**.

## Chain Rule :

If  $f(x)$  and  $g(x)$  are differentiable function, then the composite function  $(f \circ g)(x) = f(g(x))$  is also differentiable and

$$(f \circ g)'(x) = f'(g(x)) g'(x) .$$

Hard to understand? Let's rewrite :

Let  $u = g(x)$ ,  $y = f(u) = f(g(x))$ , then

Chain rule : 
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Think as : 
$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

e.g. Find the derivative of  $\sqrt{x^2+3x}$ .

$$\text{Let } u = g(x) = x^2 + 3x,$$

$$\frac{du}{dx} = 2x + 3$$

$$y = f(u) = \sqrt{u}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\text{then } f(g(x)) = \sqrt{x^2+3x}$$

$$\text{By chain rule, } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot (2x+3)$$

$$= \frac{1}{2\sqrt{x^2+3x}} \cdot (2x+3)$$

put  $u = x^2 + 3x$  back

$f'(g(x))$

$g'(x)$

differentiate  $f$   
then put back  $g(x)$

e.g. Find the derivative of  $(3x^2-2x)^{2015}$ .

$$\text{Let } u = g(x) = 3x^2 - 2x \quad \frac{du}{dx} = 6x - 2$$

$$y = f(u) = u^{2015} \quad \frac{dy}{du} = 2015 u^{2014}$$

$$\text{then } f(g(x)) = (3x^2 - 2x)^{2015}$$

$$\text{By chain rule, } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2015 u^{2014} \cdot (6x - 2)$$

$$= 2015 (3x^2 - 2x)^{2014} \cdot (6x - 2) \quad \text{put } u = 3x^2 - 2x \text{ back}$$

$$= 4030 (3x^2 - 2x)^{2014} \cdot (3x - 1)$$

Slogan: differentiate layer by layer.

Ex: By using chain rule, show that  $\frac{d}{dx} e^{ax} = ae^{ax}$

Ex: Find the derivative of  $\left(\frac{x}{x+1}\right)^2$ .

(a) By chain rule;

(b) Write  $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$ , then by quotient rule.

Ans: Both equal to  $\frac{2x}{(x+1)^3}$ .

e.g. Find the derivative of  $e^{\sqrt{x^2+1}}$ .

$$\text{1st layer } y = e^w \quad w = \sqrt{x^2+1}$$

$$\text{2nd layer } w = \sqrt{u} \quad u = x^2+1$$

$$\text{3rd layer } u = x^2+1$$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{x e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

e.g. Revisit of quotient rule.

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \frac{d}{dx} \left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx} (f(x) [g(x)]^{-1}) \\ &= \frac{df}{dx} [g(x)]^{-1} + f(x) \frac{d}{dx} [g(x)]^{-1} \quad (\text{Product rule})\end{aligned}$$

↪ Apply chain rule here

$$= \frac{df}{dx} [g(x)]^{-1} + f(x) \left\{ -[g(x)]^{-2} \frac{dg}{dx} \right\}$$

$$= \frac{\frac{df}{dx} g(x) - f(x) \frac{dg}{dx}}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$



## Differentiability :

Theorem : If  $f(x)$  is differentiable at  $x=x_0$ , then  $f(x)$  is continuous at  $x=x_0$ .

proof : By assumption ,  $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$  exists.  
Also, we know  $\lim_{\Delta x \rightarrow 0} \Delta x = 0$

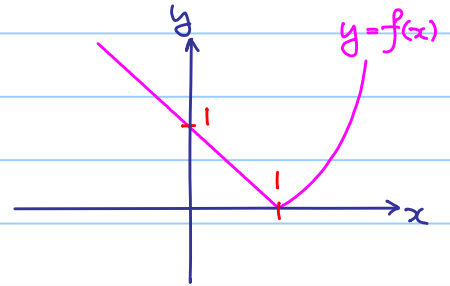
$$\begin{aligned}\lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) - f(x_0) &= \lim_{\Delta x \rightarrow 0} \left( \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right) \\ &= \left( \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right) \cdot \left( \lim_{\Delta x \rightarrow 0} \Delta x \right) \\ &= f'(x_0) \cdot 0 = 0\end{aligned}$$

both exist

$\therefore \lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) = f(x_0)$ , so  $f(x)$  is continuous at  $x=x_0$ .

However, the converse is **NOT** true.

e.g. If  $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$



$$\lim_{\Delta x \rightarrow 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{[(1+\Delta x)^2 - 1] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{2\Delta x + \Delta x^2}{\Delta x} = 2$$

↑  
(it means we are looking at  
small but positive  $\Delta x$ )

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{[1 - (1+\Delta x)] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1$$

↑  
(it means we are looking at  
small but negative  $\Delta x$ )

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} \neq \lim_{\Delta x \rightarrow 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} \Rightarrow \lim_{\Delta x \rightarrow 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} \text{ does NOT exist!}$$

$\therefore f(x)$  is NOT differentiable at  $x=1$ .

Ex:

a) Show that  $f(x)$  is continuous at  $x=1$ , i.e.  $\lim_{x \rightarrow 1} f(x) = f(1)$ .

(Therefore,  $f(x)$  is continuous at  $x=1$ , but NOT differentiable at  $x=1$  )

b) Show that  $f(x)$  is differentiable everywhere except  $x=1$ , and

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ \text{undefined} & \text{if } x = 1 \\ -1 & \text{if } x < 1 \end{cases}$$

e.g. Let  $f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Does  $f'(0)$  exist?

$$\lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2 \cos \frac{1}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \Delta x \cos \frac{1}{\Delta x}$$

$$= 0$$

By sandwich theorem

$$\text{If } x \neq 0, f'(x) = 2x \cos \frac{1}{x} + x^2 (-\sin \frac{1}{x}) \left(-\frac{1}{x^2}\right)$$

$$= 2x \cos \frac{1}{x} + \sin \frac{1}{x}$$

$\therefore f$  is a differentiable function, i.e. differentiable at every point.

Note: It is wrong to say  $f'(x) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}$ , so  $f'(0)$  does NOT exist.

$$\text{Now, } f'(x) = \begin{cases} 2x \cos \frac{1}{x} + \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Ex: Show  $\lim_{x \rightarrow 0} f'(x)$  does NOT exist ( $\Rightarrow f'(x)$  is NOT continuous at  $x=0$ )

$\therefore f$  is differentiable ("good" in some sense)

but  $f'(x)$  can be "bad".

e.g. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a **non-constant** function such that

(i)  $f$  is differentiable at some  $x_0 \in \mathbb{R}$

(ii)  $f(x+y) = f(x)f(y)$  for all  $x, y \in \mathbb{R}$ .

Show that :

a)  $f(x) \neq 0$  for all  $x \in \mathbb{R}$  and  $f(0) = 1$ .

b)  $f$  is differentiable at every  $x \in \mathbb{R}$  and  $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$ .



a) If  $f(a) = 0$  for some  $a \in \mathbb{R}$

then for any  $x \in \mathbb{R}$ , we have

$$f(x) = f(x-a+a) = f(x-a)f(a) = 0$$

i.e.  $f(x)$  is constant zero (Contradict to the assumption)

$$\therefore f(x) \neq 0 \quad \forall x \in \mathbb{R}.$$

Putting  $x = y = 0$ ,

$$f(0+0) = f(0)f(0)$$

$$f(0) = [f(0)]^2$$

$$f(0) = 1 \quad \text{or} \quad 0 \quad (\text{rejected})$$

b)  $f$  is differentiable at  $x_0$

$$\begin{aligned}\Rightarrow f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0) f(\Delta x) - f(x_0)}{\Delta x}\end{aligned}$$

$$\therefore \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x) - f(x_0)}{\Delta x} = \frac{f'(x_0)}{f(x_0)} \quad (\because f(x_0) \neq 0)$$

$$\begin{aligned}\text{Now, } \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x) f(\Delta x) - f(x)}{\Delta x} \\ &= f(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x) - f(x_0)}{\Delta x} \\ &= \frac{f'(x_0)}{f(x_0)} f(x)\end{aligned}$$

$\therefore f$  is differentiable at every  $x \in \mathbb{R}$  and  $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$ .

(In fact,  $f(x) = e^{ax}$  for some non-zero constant  $a$ .)

Ex: Let  $f$  be a differentiable function such that

$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \forall x, y \in \mathbb{R}.$$

a) Show that  $f'(0) = \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$ .

b) Hence, show that  $f'(x) = f'(0) + 3x^2$ .

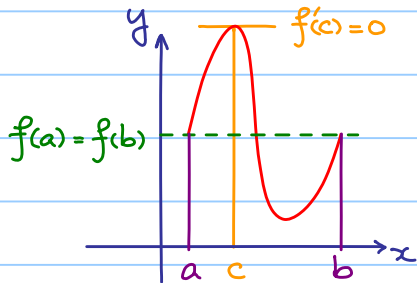
(In fact,  $f(x) = f'(0)x + x^3$ .)

## Rolle's Theorem

If  $f$  is a function such that :

- 1)  $f$  is continuous on  $[a, b]$
- 2)  $f$  is differentiable on  $(a, b)$
- 3)  $f(a) = f(b)$

then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

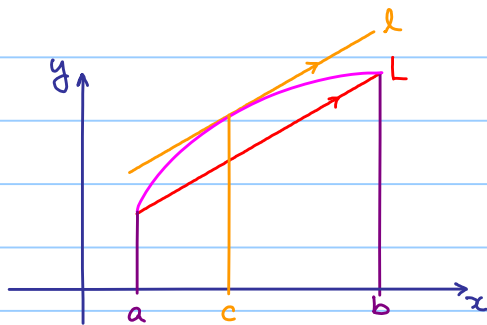


## Mean Value Theorem

If  $f$  is a function such that :

- 1)  $f$  is continuous on  $[a, b]$
- 2)  $f$  is differentiable on  $(a, b)$

then there exists  $c \in (a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$



slope of  $l$

slope of  $L$ .

proof: Let  $g(x) = f(x)(b-a) - x[f(b) - f(a)]$

Check: 1)  $g$  is continuous on  $[a, b]$

2)  $g$  is differentiable on  $(a, b)$

3)  $g(a) = g(b) = b f(a) - a f(b)$

Apply Rolle's Theorem to  $g$ , the result follows.

e.g. A vehicle is speeding on a highway if its speed  $\geq 120$  km/h (at some moment)  
If the length of the highway is 30 km and if Kelvin only spent 15 minutes on  
the highway. Should he be arrested?

Theorem :

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable and  $f'(x) = 0 \quad \forall x \in \mathbb{R}$ ,  
then  $f(x)$  is a constant function.

proof: Fix  $x_0 \in \mathbb{R}$ , let  $x \in \mathbb{R} \setminus \{x_0\}$

If  $x > x_0$ , note  $f$  is differentiable everywhere (in particular, on  $(x_0, x)$ )

$\Rightarrow f$  is continuous everywhere (in particular, on  $[x_0, x]$ )

Apply MVT,  $\exists c \in (x_0, x)$  such that

$$f(x_0) - f(x) = \underbrace{f'(c)}_0 (x - x_0) = 0$$

||  
0 by assumption.

$$\text{i.e. } f(x) = f(x_0) \quad \forall x > x_0$$

We have similar result if  $x < x_0$ , the result follows.

e.g. Let  $f(x) = \cos^2 x + \sin^2 x$

$$f'(x) = -2\cos x \sin x + 2\sin x \cos x = 0$$

$\therefore \cos^2 x + \sin^2 x$  is a constant.

In particular,  $f(0) = 1$ , so  $f(x) = \cos^2 x + \sin^2 x = 1$



Theorem :

If  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are differentiable and  $f'(x) = g'(x) \forall x \in \mathbb{R}$ ,  
then  $f(x) = g(x) + C$ , where  $C$  is a constant.

proof: Let  $h(x) = f(x) - g(x)$ .

Then  $h'(x) = f'(x) - g'(x) = 0$

$\therefore h(x) = C$ , where  $C$  is a constant. i.e.  $f(x) = g(x) + C$ .

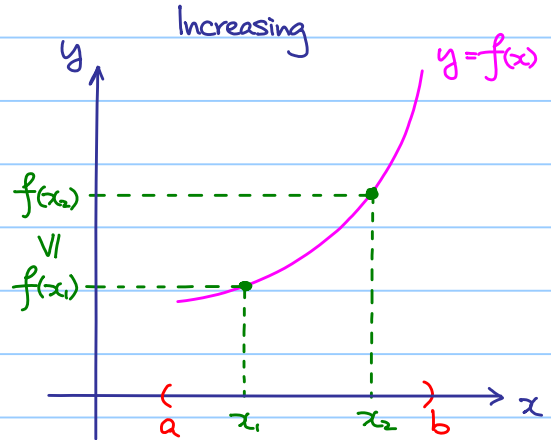
Next, we are going to discuss how differentiation helps to find maximum / minimum points of a function.

Firstly, we make some preparations:

## Increasing / Decreasing Functions

If  $f(x)$  is a function such that for all  $x_1, x_2$  with  $a < x_1 < x_2 < b$ , we have  $f(x_1) \leq f(x_2)$  ( $f(x_1) \geq f(x_2)$ ), then  $f(x)$  is called an increasing (a decreasing) function on  $(a, b)$ .

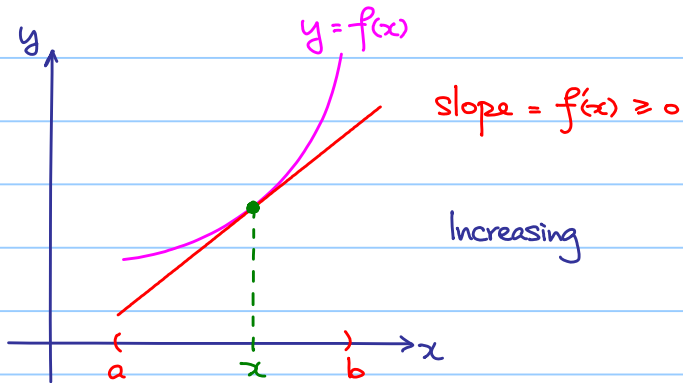
Roughly speaking:  
The larger  $x$  we input  
the larger  $y$  we get!



† If we have strict inequality, it is called a strictly increasing (decreasing) function on  $(a, b)$ .

Theorem:

If  $f(x)$  is differentiable on  $(a, b)$  and  $f'(x) \geq 0$  ( $f'(x) \leq 0$ ) for all  $x \in (a, b)$ , then  $f(x)$  is increasing (decreasing) on  $(a, b)$ .



$\dagger\dagger$  If we have strict inequality,  $f(x)$  is a strictly increasing (decreasing) function on  $(a, b)$ .

proof: If  $a < x_1 < x_2 < b$ ,

apply MVT to  $f$  on  $[x_1, x_2]$ ,

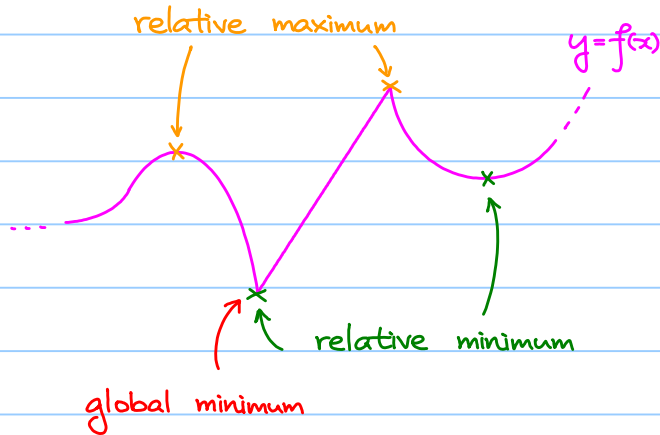
$$\exists c \in (x_1, x_2) \text{ such that } f(x_2) - f(x_1) = \underbrace{f'(c)}_{\substack{> \\ 0}} \underbrace{(x_2 - x_1)}_{> 0} \geq 0$$

By assumption

## Relative / Global Extrema :



Idea :



Note : No global maximum  
in this case .

$f$  has a **global maximum (resp. minimum)** point at  $a$  if  
 $f(x) \leq f(a)$  (resp.  $f(x) \geq f(a)$ ) for all  $x$  in the domain of  $f$ .

$f$  has a **relative maximum (resp. minimum)** point at  $a$  if  
 $f(x) \leq f(a)$  (resp.  $f(x) \geq f(a)$ ) for all  $x$  in a neighborhood of  $a$ .

e.g. Number of days of using drug :  $x$

Life of a fish :  $T$  (weeks) which is estimated by

$$T(x) = -5x^2 + 80x - 120$$

$$T'(x) = -10x + 80$$

$$T'(x) > 0$$

$$-10x + 80 > 0$$

$$x < 8$$

$$T'(x) < 0$$

$$-10x + 80 < 0$$

$$x > 8$$

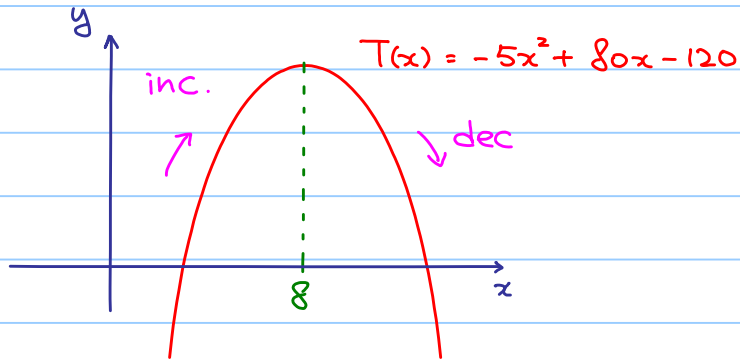
$\therefore T(x)$  is strictly increasing when  $x < 8$  and

$T(x)$  is strictly decreasing when  $x > 8$ .

Not hard to understand why  $T(x)$  attains maximum when  $x = 8$

and maximum life of a fish =  $T(8) = 200$  (weeks)





Note :  $T'(8) = 0$ .

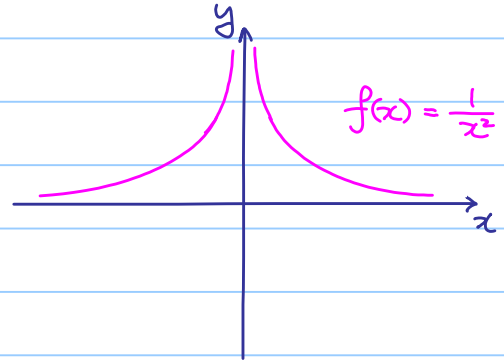
Remark : Verify the above result by completing square .

e.g. Let  $f(x) = \frac{1}{x^2}$ ,  $x \neq 0$

$$f'(x) = \frac{-2}{x^3}$$

$$f'(x) > 0 \quad \text{if } x < 0$$

$$f'(x) < 0 \quad \text{if } x > 0$$



$\therefore f(x)$  is strictly increasing when  $x < 0$

$f(x)$  is strictly decreasing when  $x > 0$

However,  $f(0)$  is NOT well-defined, so there is NO maximum point.

e.g. Let  $f(x) = \sqrt{|x|}$

Rewrite:

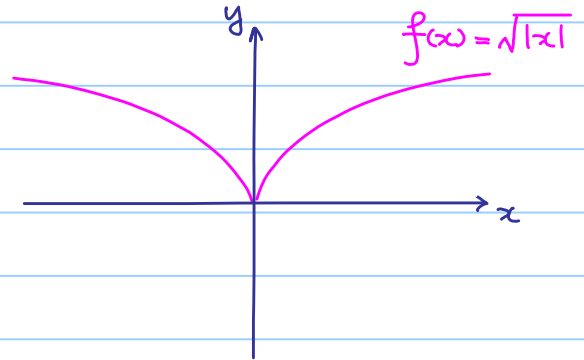
$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

If  $x > 0$ ,  $f(x) = \sqrt{x}$ , then  $f'(x) = \frac{1}{2\sqrt{x}} > 0$

If  $x < 0$ ,  $f(x) = \sqrt{-x}$ , then  $f'(x) = -\frac{1}{2\sqrt{-x}} < 0$

$\therefore f(x)$  is strictly increasing when  $x > 0$

$f(x)$  is strictly decreasing when  $x < 0$



However,  $\lim_{\Delta x \rightarrow 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{\sqrt{\Delta x}}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{1}{\sqrt{\Delta x}}$  which does NOT exist,

$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$  does NOT exist

$\Rightarrow f'(0)$  does NOT exist

but as we can see  $f$  still attains minimum at  $x=0$ .

$\therefore$  Solving  $f'(x) = 0$  to find max/min is NOT enough.

Exact statement :

### 1st Derivative Check :

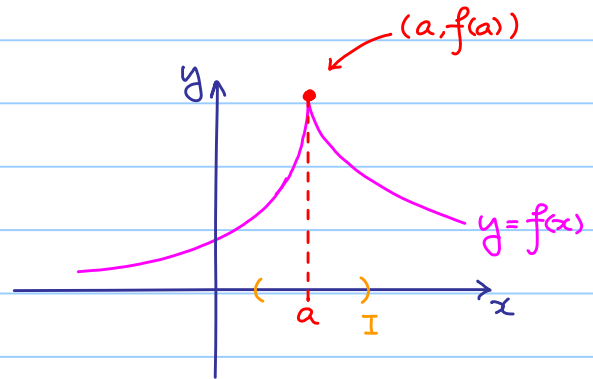
Suppose  $f(x)$  is continuous at  $x=a$  and differentiable on some neighborhood  $I$  containing  $a$ , except possibly at  $x=a$  itself.

If  $f'(x) \geq 0$  for all  $x$  in  $I$  with  $x < a$ , and

$f'(x) \leq 0$  for all  $x$  in  $I$  with  $x > a$ ,

then  $(a, f(a))$  is a relative maximum.

(Similar for relative minimum.)



(Remember the slogan : Change sign of  $f'(x)$  at  $x=a$ )

proof: If  $x \in I$  and  $x < a$ ,

apply MVT to  $f$  on  $[x, a]$ ,

$$\exists c \in (x, a) \text{ such that } f(a) - f(x) = \underbrace{f'(c)}_{\substack{\forall \\ 0}} \underbrace{(a-x)}_{\substack{\forall \\ 0}} \geq 0$$

By assumption

$$\therefore f(x) \leq f(a) \quad \forall x \in I \text{ with } x < a$$

Similarly, we have  $f(x) \leq f(a) \quad \forall x \in I$  with  $x > a$

$$\therefore f(x) \leq f(a) \quad \forall x \in I.$$