THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G University Mathematics 2014-2015
Test 1, 10 Feb, 2015

- Time allowed: 45 minutes
- Answer all questions.
- Show your work clearly and concisely in your answer book.
- Write down your name and student ID number on the front page of your answer book.
- You are allowed to use a calculator in this test.

1. Evaluate each of the following limits.

   (a) $\displaystyle\lim_{n\to\infty} \frac{5n^2-1}{n^2-3n+2}$.

   (b) $\displaystyle\lim_{n\to\infty} (1+\frac{1}{2n})^{n+1}$.

(12 points)

2. By using sandwich theorem, find the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^3+1}} + \frac{1}{\sqrt[3]{n^3+2}} + \cdots + \frac{1}{\sqrt[3]{n^3+n}}.$$

(15 points)

3. Evaluate each of the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{x}$.

   (b) $\displaystyle\lim_{x\to+\infty} \frac{e^{x+1}+e^{-(x+1)}}{e^{x-1}-e^{-(x+1)}}$.

   (c) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{4x^2-x+1}}$.

(18 points)

4. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$.

   (a) Prove that $f$ is differentiable at $x = 0$ and find $f'(0)$.

   (b) Is $f'(x)$ differentiable at $x = 0$?

(20 points)

5. By using mean value theorem, prove that for any $x > y > 0$,

$$ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y),$$

where $n$ is a natural number.

(15 points)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

- $f(x+y) = f(x)f(y)$ for all real numbers $x$ and $y$;
- $1 + x \leq f(x) \leq 1 + xf(x)$ for all real numbers $x$.

(a) Show that

(i) $f(0) = 1$,

(ii) $f(x) > 1$ for $x > 0$,

(iii) $f(x) > 0$ for all real number $x$.

Hence, deduce that $f(x)$ is strictly increasing, that means if $a > b$, then $f(a) > f(b)$.

(b) Show that if $h < 1$, we have

$$1 + h \leq f(h) \leq \frac{1}{1-h}.$$

Hence, show that $f(x)$ is continuous at $x = 0$.

(c) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

(20 points)