e.g. Let $y=\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$. Find $\frac{dy}{dx}$.

$$y=\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$$

$$\ln y = 5x+\frac{1}{3}\ln(x^2+1)-4\ln(3x^2+1)$$

Ex: $\vdots$

Ans: $\frac{dy}{dx}=\left[5+\frac{2x}{3(x^2+1)}-\frac{24x}{3x^2+1}\right]\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$

e.g. Let $y=x^x$, $x>0$. Find $\frac{dy}{dx}$.

Note: The power is NOT a constant, we cannot use the formular $\frac{d}{dx}x^n=nx^{n-1}$.

$$y=x^x$$

$$\ln y=\ln x^x=x\ln x$$

differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx}=\ln x+x\cdot\frac{1}{x}$$

$$=\ln x+1$$

$$\frac{dy}{dx}=(\ln x+1)y$$

$$=(\ln x+1)x^x$$

e.g. (2nd derivative)

Suppose $x^3+y^3-3xy=0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$x^3+y^3-3xy=0$$

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$

differentiate both sides with respect to x again.

$$\frac{d^2y}{dx^2}=\frac{\frac{dy}{dx}(y^2-x)-(y-x^2)(2y\frac{dy}{dx}-1)}{(y^2-x)^2}$$

Sub. $\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$ back to express $\frac{d^2y}{dx^2}$ in terms of x and y only, if you want.

Nightmare!

Application of Differentiation :

e.g. Bacteria in closed environment with nutrition

What happens :

- Number of bacteria increases as plenty of nutrition at the beginning.
- Number of bacteria decreases as nutrition is no longer sufficient to support a large number of bacteria.

Suppose the number of bacteria t hours after the start of the experiment is modeled by the function

$$N(t) = \frac{10t+5}{e^{t+1}} \quad \text{(thousand)} \quad , \; t \geq 0 .$$

① Number of bacteria at the beginning

$= N(0) = 5/e \approx 1.84$ (thousand)

② $N'(t) = \dfrac{10e^{t+1} - (10t+5)e^{t+1}}{e^{2(t+1)}}$

$= \dfrac{-10t+5}{e^{t+1}}$

$N'(t) > 0 \Leftrightarrow -10t+5 > 0 \Leftrightarrow t < 0.5$

$N'(t) < 0 \Leftrightarrow -10t+5 < 0 \Leftrightarrow t > 0.5$

1st derivative check $\Rightarrow$ $N(t)$ attains max. when $t = 0.5$

Max. number of bacteria $= N(0.5) = 10/e^{1.5} \approx 2.23$ (thousand)

③ $\lim\limits_{t \to +\infty} N(t) = \lim\limits_{t \to +\infty} \dfrac{1}{e} \dfrac{10t+5}{e^{t}}$

$= 0$

Recall: If $p(x)$ is a polynomial, then $\lim\limits_{x \to +\infty} \dfrac{p(x)}{e^{x}} = 0$.

i.e. Extinct eventually !

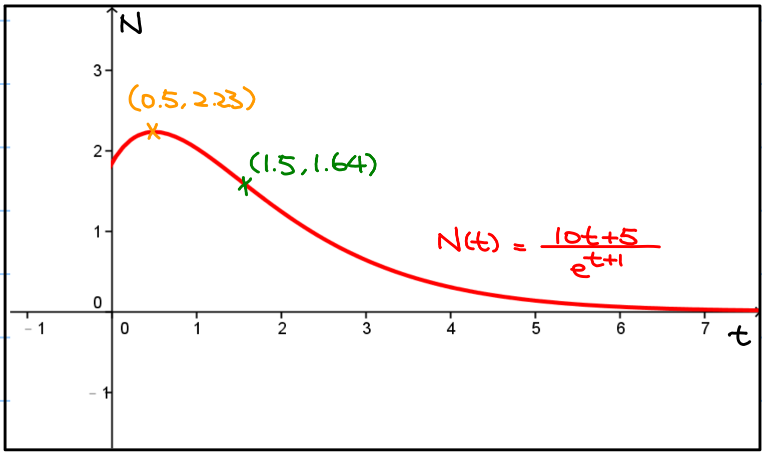
④ $N''(t) = \dfrac{-10e^{t+1} - (-10t+5)e^{t+1}}{e^{2(t+1)}}$

$= \dfrac{10t-15}{e^{t+1}}$

$N''(t) > 0 \Leftrightarrow 10t - 15 > 0 \Leftrightarrow t > 1.5$

$N''(t) < 0 \Leftrightarrow 10t - 15 < 0 \Leftrightarrow t < 1.5$

$\therefore$ point of inflection $= (1.5, N(1.5)) \approx (1.5, 1.64)$



On the other hand, $N'(t)$ attains min when $t = 1.5$

$N'(1.5) = -10/e^{1.5} = -2.23$

i.e. Number of bacteria decreases most rapidly at $t = 1.5$ and

decreasing rate at $t = 1.5$ is 2.23 thousand / hour.

e.g. A manager of a company, determines that t months after initiating an advertisting compaign, the number of products will be sold is estimated by

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5 \quad \text{(thousand)} \quad , \; t \geq 0.$$

a) Find $P'(t)$ and $P''(t)$.

b) At what time will sales be maximized? What is the maximum level of sales?

c) The manager plans to terminate the advertising compaign when the sales rate is minimized. When does it occur?

a) Direct computation:

$$P(t) = \frac{3}{t+2} - \frac{12}{(t+2)^2} + 5$$

$$P'(t) = -\frac{3}{(t+2)^2} + \frac{24}{(t+2)^3} = \frac{18-3t}{(t+2)^3}$$

$$P''(t) = \frac{6}{(t+2)^3} - \frac{72}{(t+2)^4} = \frac{6t-60}{(t+2)^4}$$

(b) Solve $P'(t) > 0$

$$\frac{18-3t}{(t+2)^3} > 0$$

$$18-3t > 0 \quad (\because t \geq 0,\ t+2>0)$$

$$t < 6$$

$P'(t) < 0$

$$\frac{18-3t}{(t+2)^3} < 0$$

$$18-3t < 0$$

$$t > 6$$

($P(t)$ is strictly increasing when $t<6$ and strictly decreasing when $t>6$, $P(t)$ is continuous at $t=6$.)

$\therefore P(t)$ attains maximum when $t=6$. (By 1st derivative check.)

OR: (By observation, $P(t)$ can be differentiated infinitely many times, so if $P(t)$ attains maximum / minimum at $t=t_0$, we must have $P'(t_0)=0$, that's why we consider the equation $P'(t)=0$.)

$$P'(t) = 0$$

$$\frac{18-3t}{(t+2)^3} = 0$$

$$t = 6$$

(At this moment, we only know $(6, P(6))$ is a stationary point.)

$$P''(6) = -\frac{24}{8^4} < 0$$

$\therefore P(t)$ attains maximum when $t=6$. (By 2nd derivative check.)

Maximum sales level $= P(6) = \frac{83}{16}$

(c) (In fact, we want to minimize $P'(t)$ now!

We apply 1st derivative check to $P'(t)$, i.e. look at $P''(t)$.)

Solve $P''(t) > 0$ $P''(t) < 0$

$\frac{6t-60}{(t+2)^4} > 0$ $\frac{6t-60}{(t+2)^4} < 0$

$6t-60 > 0$ $6t-60 < 0$

$t > 10$ $t < 10$

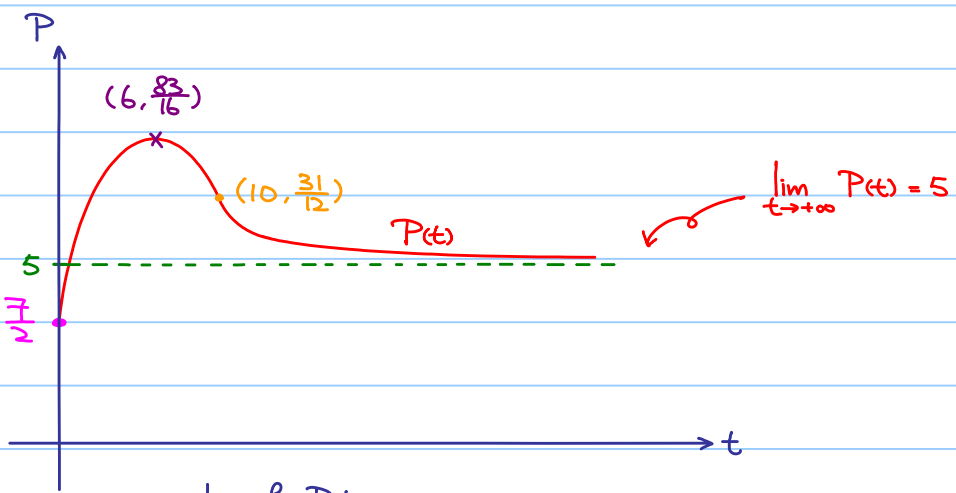
$\therefore$ $P'(t)$ attains minimum when $t = 10$. (By 1st derivative check.)

(Note: $(10, P(10))$ is a point of inflection.)

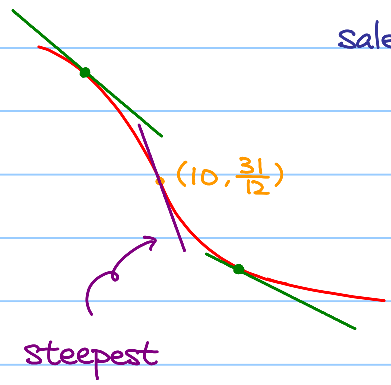
OR: $P'''(t) = -\frac{18}{(t+2)^4} + \frac{288}{(t+2)^5} = \frac{252-18t}{(t+2)^5}$

$P'''(10) = \frac{72}{12^5} > 0$

$\therefore$ $P'(t)$ attains minimum when $t = 10$. (By 2nd derivative check.)



graph of $P(t)$

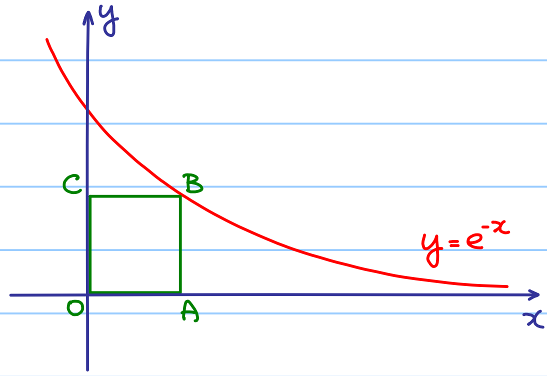


sales rate at t = $P'(t)$

= slope of the tangent line at $(t, P(t))$

Meaning of minimizing $P'(t)$ in part (c).

e.g. OABC is a rectangle inscribed in the region bounded by the positive coordinate axes and the curve $y=e^{-x}$. Find the maximum area of the rectangle.



Maximize a function!

Dependent variable : Area of OABC, A

Independent variable : x

$$\text{Area of } OABC = OA \times AB$$

$$A = xe^{-x} \qquad x \geq 0$$

$$\frac{dA}{dx} = e^{-x} - xe^{-x}$$

$$= e^{-x}(1-x)$$

$\frac{dA}{dx} > 0$	$\frac{dA}{dx} < 0$
$e^{-x}(1-x) > 0$	$e^{-x}(1-x) < 0$
$1-x > 0$	$1-x < 0$
$1 > x$	$1 < x$

$\therefore$ A attains maximum when $x=1$.

Maximum area of OABC $= A(1) = 1 \cdot e^{-1} = e^{-1}$

Remark : Most Important issue :

1) identifying dependent and independent variable

2) setting up an equation between them

Relative Rates

Suppose x and y are variables related by an equation, but both of them can further be regarded as functions of a third variable t.

(i.e. $x(t)$ and $y(t)$)

(Often : t = time)

Then Implicit differentiation helps to give a relation between $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

e.g. Relation of pollution and population of fish.

Level of popular*t* = x parts per million (ppm)

Number of fish = F

Given $F = \frac{32000}{3+\sqrt{x}}$

When there are 4000 fish left in the lake,

the population is increasing at the rate of 1.4 ppm/year.

At what rate is the fish population changing at this time?

time : t (years)

$F = 4000$

$\frac{dx}{dt} = 1.4$ (increasing, $\frac{dx}{dt} > 0$; decreasing, $\frac{dx}{dt} < 0$)

$\frac{dF}{dt} = ?$ when $\frac{dx}{dt} = 1.4$, $F = 4000$

Idea : Apply implicit differentiation to the equation $F = \frac{32000}{3+\sqrt{x}}$ and differentiate with respect to t

$$\frac{dF}{dt} = \frac{d}{dt}\left(\frac{32000}{3+\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{32000}{3+\sqrt{x}}\right)\frac{dx}{dt}$$ (Apply chain rule)

$$\frac{dF}{dt} = \frac{-16000}{\sqrt{x}(3+\sqrt{x})^2}\frac{dx}{dt}$$

$\frac{dx}{dt} = 1.4$ Oops, $x = ?$

Recall: $F = \dfrac{32000}{3+\sqrt{x}}$, when $x = 4000$

$$4000 = \frac{32000}{3+\sqrt{x}}$$

$$x = 25$$

$$\frac{dF}{dt} = \frac{-16000}{\sqrt{x}(3+\sqrt{x})^2}\frac{dx}{dt} = \frac{-16000}{\sqrt{25}(3+\sqrt{25})^2} \times 1.4 = -70 \quad \text{(fish per year)}$$

Note: Reasonable!

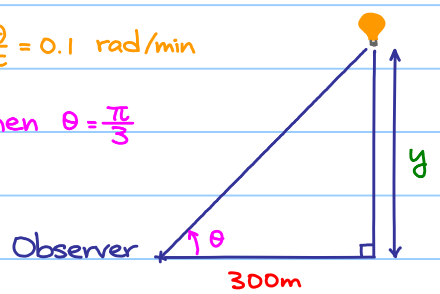
$\frac{dx}{dt} = 1.4 > 0$, i.e. pollutant is increasing.

$\frac{dF}{dt} = -70 < 0$, i.e. population of fish is decreasing.

e.g. A hot air balloon rising straight up from a level field is tracked by an observer 300m from the liftoff point. At the moment the observer's elevation angle is $\pi/3$, the angle is increasing at the rate 0.1 rad/min. How fast is the balloon rising at that moment?

$\frac{d\theta}{dt} = 0.1$ rad/min

when $\theta = \frac{\pi}{3}$



$\frac{dy}{dt} = ?$

when $\theta = \frac{\pi}{3}$

Setting up an equation between y and θ:

$$y = 300 \tan\theta$$

differentiate both sides with respect to t,

$$\frac{dy}{dt} = 300\sec^2\theta \frac{d\theta}{dt}$$

When $\theta = \pi/3$,

$$\frac{dy}{dt} = 300\sec^2\frac{\pi}{3} \cdot 0.1$$

$$= 120 \text{ m/min}$$