Math4230 Exercise 9 Solution

1. Suppose x^* is a local minimum which is not global. Then there exists $\overline{x}$ such that $f(\overline{x}) < f(x^*)$. For $\alpha \in (0,1)$,
$$f(\alpha x^* + (1-\alpha)\overline{x}) \leq \alpha f(x^*) + (1-\alpha) f(\overline{x}) < f(x^*)$$
This contradicts the local minimality of x^*.

2. Suppose x^* is a global minimum of f. Let $x \in \mathbb{R}^n$ and $x \neq x^*$. Then
$$f(x^*) \leq f(\frac{1}{2}(x + x^*)) < \frac{1}{2}(f(x) + f(x^*)).$$
Hence, $f(x^*) < f(x)$. So x^* is the unique global minimum.

3. (a) Suppose x^* minimizes f over X. Let $y \in Y$. Then for all $x \in X$,
$$f_c(x^*) = f(x^*) \leq f(x) \leq f(y) + L||y-x|| < f(y) + c||y-x||$$
Taking infnmum over X, we have $f_c(x^*) \leq f_c(y)$.
So x^* minimizes f_c over Y.

   (b) Suppose $x^* \notin X$ minimizes f_c over Y. Since X is closed, there exists $\tilde{x}$ such that $||\tilde{x} - x^*|| = \inf_{\overline{x} \in X} ||\overline{x} - x^*||$. Then
$$f_c(x^*) = f(x^*) + c||\tilde{x} - x^*|| > f(x^*) + L||\tilde{x} - x^*|| \geq f(\tilde{x}) = f_c(\tilde{x})$$
This contradicts the minimality of x^*. Hence, $x^* \in X$.