Math4230 Exercise 8 Solution

1. Suppose x^* is a minimizer of f, then $f(y) \geq f(x^*) = f(x^*) + \langle 0, y - x^* \rangle$ for all y. Hence $0 \in \partial f(x^*)$.
   Conversely, suppose $0 \in \partial f(x^*)$. Then $f(y) \geq f(x^*) + \langle 0, y - x^* \rangle$ for all y. Hence, x^* is a minimizer of f.

2. Suppose $g \in \partial f(x)$. Let y be such that $y < x$. Then $f(y) \geq f(x) + g(y-x)$
   Since $f(y) \leq f(x)$, we have $g(x-y) \geq 0$

3. (a) Since $f'(x;0) = 0$, the equality holds when $\lambda = 0$.
   So assume $\lambda > 0$,
   $$f'(x;\lambda y) = \inf_{\alpha>0} \frac{f(x+\alpha\lambda y) - f(x)}{\alpha} = \lambda \inf_{\alpha>0} \frac{f(x+\alpha\lambda y) - f(x)}{\alpha\lambda}$$
   Hence $f'(x;\lambda y) = \lambda f'(x;y)$ by considering $\beta = \alpha\lambda$.

   (b) Let y_1, y_2 be two points.
   Let $\lambda \in (0,1), y_\lambda = \lambda y_1 + (1-\lambda) y_2$.
   By convexity of f, $f(x + \alpha y_\lambda) \leq \lambda f(x + \alpha y_1) + (1-\lambda) f(x + \alpha y_2)$.
   Hence,
   $$\frac{f(x+\alpha y_\lambda) - f(x)}{\alpha} \leq \lambda \frac{f(x+\alpha y_1) - f(x)}{\alpha} + (1-\lambda) \frac{f(x+\alpha y_2) - f(x)}{\alpha}$$
   Since the difference quotient is increasing as α increases, we can replace α by some $\alpha_1, \alpha_2 \geq \alpha$ on the right hand side. So
   $$\frac{f(x+\alpha y_\lambda) - f(x)}{\alpha} \leq \lambda \frac{f(x+\alpha_1 y_1) - f(x)}{\alpha_1} + (1-\lambda) \frac{f(x+\alpha_2 y_2) - f(x)}{\alpha_2}$$
   Taking infimum over α, and then α_1, α_2 we have
   $$f'(x;y_\lambda) \leq \lambda f'(x;y_1) + (1-\lambda) f'(x;y_2)$$

   (c) $f'(x; \frac{1}{2}y + \frac{1}{2}(-y)) = f'(x;0) = 0$.
   But $f'(x;.)$ is convex, so
   $$0 \leq \frac{1}{2} f'(x,y) + \frac{1}{2} f'(x;-y)$$
   Hence $-f'(x;-y) \leq f'(x;y)$.