Math4230 Exercise 5 Solution

1. $\overline{\text{ri}(C)} \subset \overline{C}$ since $\text{ri}(C) \subset C$.
   Conversely, suppose $x \in \overline{C}$.
   Let $\overline{x} \in \text{ri}(C)$. Consider $x_k = \frac{1}{k}\overline{x} + (1 - \frac{1}{k})x$.
   By the line segment property, each $x_k \in \text{ri}(C)$. Also, $x_k \to x$. Therefore, $x \in \overline{\text{ri}(C)}$.

2. We first prove that $\text{ri}(C) = \text{ri}(\overline{C})$. $\text{ri}(C) \subset \text{ri}(\overline{C})$ follows from the definition and the fact that $\text{aff}(C) = \text{aff}(\overline{C})$.(Try to show this)
   Conversely, suppose $x \in \text{ri}(\overline{C})$. Suppose $\overline{x} \in \text{ri}(C)$. (which exists since $\text{ri}(C)$ is nonempty)
   We may assume $x \neq \overline{x}$. Then by Prolongation lemma, $y = x + \gamma(x - \overline{x}) \in \overline{C}$, for some $\gamma > 0$.
   Then $x = \frac{\gamma}{1+\gamma}\overline{x} + \frac{1}{1+\gamma}y$. By Line Segment Property, $x \in \text{ri}(C)$.
   Now, since $\overline{C_1} = \overline{C_2}$, $\text{ri}(\overline{C_1}) = \text{ri}(\overline{C_2})$. Hence, $\text{ri}(C_1) = \text{ri}(C_2)$.

3. (a) $C_1 = \{(x, y) \mid 0 \leq x \leq 1,\ y = 0\}$
   $C_2 = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$
   (b) Let $x \in \text{ri}(C_1)$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap \text{aff}(C_1) \subset C_1$.
   But $\text{aff}(C_1) = \text{aff}(C_2)$. So $B(x, \epsilon) \cap \text{aff}(C_2) \subset C_1 \subset C_2$.
   Hence $x \in \text{ri}(C_2)$.

4. Let $x^* \in X^* \cap \text{ri}(X)$. Let $x \in X$.
   By Prolongation lemma, $y = x^* + \gamma(x^* - x) \in X$.
   So $x^* = \frac{\gamma}{1+\gamma}x + \frac{1}{1+\gamma}y$. Since f is concave, we have

$$f(x^*) \geq \frac{\gamma}{1+\gamma}f(x) + \frac{1}{1+\gamma}f(y) \geq \frac{\gamma}{1+\gamma}f(x^*) + \frac{1}{1+\gamma}f(x^*) = f(x^*)$$

   since $f(x) \geq f(x^*)$, $f(y) \geq f(x^*)$.
   So we must have equality. In particular, $f(x) = f(x^*)$. This holds for any $x \in X$. Hence, f must be constant.