Math4230 Exercise 4 Solution

1. Suppose f is not constant. Then there exists x, y such that $f(x) > f(y)$. Then for $\lambda \in (0,1)$,
$$f(x) = f\left(\lambda \frac{x-(1-\lambda)y}{\lambda} + (1-\lambda)y\right) \leq \lambda f\left(\frac{x-(1-\lambda)y}{\lambda}\right) + (1-\lambda)f(y)$$
So
$$f\left(\frac{x-(1-\lambda)y}{\lambda}\right) \geq \frac{f(x)-(1-\lambda)f(y)}{\lambda} = \frac{f(x)-f(y)}{\lambda} + f(y)$$
Since $f(x) > f(y)$, this tends to ∞ as $\lambda \to 0^+$. This contradicts the fact that f is bounded. Hence, f is constant.

2. Note that $x_2 = \frac{x_3-x_2}{x_3-x_1}x_1 + \frac{x_2-x_1}{x_3-x_1}x_3$. Then by convexity of f, we have
$$f(x_2) \leq \frac{x_3-x_2}{x_3-x_1}f(x_1) + \frac{x_2-x_1}{x_3-x_1}f(x_3)$$
Also,
$$f(x_2) = \frac{x_3-x_2}{x_3-x_1}f(x_2) + \frac{x_2-x_1}{x_3-x_1}f(x_2)$$
Hence,
$$\frac{x_3-x_2}{x_3-x_1}(f(x_2)-f(x_1)) \leq \frac{x_2-x_1}{x_3-x_1}(f(x_3)-f(x_2))$$
Therefore,
$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

3. (a) Suppose f is quasiconvex.
   Let $x, y \in V_a$, $\lambda \in [0,1]$.
$$f(\lambda x + (1-\lambda)y) \max\{f(x), f(y)\} \leq a$$
   Hence, V_a is convex for all a.
   Suppose V_a is convex for all a. Let $\lambda \in [0,1]$.
   Let $m := \max\{f(x), f(y)\}$. Then, $x, y \in V_m$.
   Since V_m is convex, $\lambda x + (1-\lambda)y \in V_m$. So $f\lambda x + (1-\lambda)y) \leq m$.
   Hence f is quasiconvex.

   (b) Since convexity implies V_a is convex for all a. A convex function is quasiconvex.
   The converse is not true. Consider $f(x) = \ln x$.

4. Suppose all level sets of f are compact. Suppose $\{x_k\}$ is a sequence with $||x_k|| \to \infty$. Suppose $f(x_k) \not\to \infty$. Then there exists subsequence x_{k_j} such that $f(x_{k_j})$ is bounded by α for some α. Then $\{x_{k_j}\} \subset V_\alpha$. This contradicts the compactness of V_α. Hence, f is coercive.
Conversely, suppose f is coercive. Suppose V_α is not compact for some α. Since f is continuous, V_α must be closed, this means V_α is not bounded. Hence, there exists a sequence $\{x_k\} \subset V_\alpha$ such that $||x_k|| \to \infty$. This contradicts the coercivity of f since $f(x_k) \leq \alpha$.