Math4230 Exercise 3 Solution

1. Let $x, y \in C$, and let $\alpha \in [0,1]$. Then

$$\begin{aligned} h(\alpha x+(1-\alpha)y) &= g(f(\alpha x+(1-\alpha)y) \\ &= g(f_1(\alpha x+(1-\alpha)y), ..., f_m(\alpha x+(1-\alpha)y)) \\ &\leq g(\alpha f_1(x)+(1-\alpha)f_1(y), ..., \alpha f_m(x)+(1-\alpha)f_m(y) \\ &= g(\alpha(f_1(x), ..., f_m(x))+(1-\alpha)(f_1(y), ..., f_m(y))) \\ &\leq \alpha g(f_1(x), ..., f_m(x))+(1-\alpha)g(f_1(y), ...f_m(y)) \\ &= \alpha g(f(x))+(1-\alpha)g(f(y)) \\ &= \alpha h(x)+(1-\alpha)h(y) \end{aligned}$$

2. (a) $z'\nabla^2 f_1(x)z = \frac{1}{(e^{x_1}+...+e^{x_n})^2}\sum_{i=1}^n\sum_{j=1}^n e^{x_i+x_j}(z_i-z_j)^2 \geq 0,\ \forall z \in \mathbb{R}^n$. Hence the Hessian of f_1 is positive semidefinite at all $x \in \mathbb{R}^n$ and f_1 is convex.
   (b) Let $f(x) = ||x||,\ g(t) = t^p$. Then use the result of (1).
   (c) Let $f(x) = x'Ax,\ g(t) = e^t$. Then use the result of (1).

3. a) We show that f is strongly convex iff

$$f(y) \geq f(x) + \langle \nabla f(x), y-x\rangle + \frac{\alpha}{2}||y-x||^2,\ \forall x, y$$

Suppose f is strongly convex. Consider $g(t) = f(x+t(y-x)$. Then $g'(t) = \langle \nabla f(x+t(y-x)), (y-x)\rangle$.
Then

$$\begin{aligned} & f(y)-f(x)-\langle \nabla f(x), y-x\rangle \\ &= g(1)-g(0)-\langle \nabla f(x), y-x\rangle \\ &= \int_0^1 \langle \nabla f(x+t(y-x)), y-x\rangle - \langle \nabla f(x), y-x\rangle dt \\ &= \int_0^1 \langle \nabla f(x+t(y-x)) - \nabla f(x), y-x\rangle dt \\ &\geq \int_0^1 \alpha t||y-x||^2 dt \\ &= \frac{\alpha}{2}||y-x||^2 \end{aligned}$$

The other direction is simple.
For $x \neq y,\ ||y = x|| \neq 0$. Hence, the above shows that f is strictly convex if f is strongly convex.
b) Consider $g(x) = f(x) - \frac{\alpha}{2}||x||^2$. $\nabla g(x) = \nabla f(x) - \alpha x$.
g is convex iff

$$g(y) \geq g(x) + \langle \nabla g(x), y-x\rangle, \forall x, y$$

This is also equivalent to

$$f(y) \geq f(x) + \langle \nabla f(x), y-x\rangle + \frac{\alpha}{2}||y-x||^2,\ \forall x, y$$

Hence f is strongly convex iff g is convex.
Note that $\nabla^2 g(x) = \nabla^2 f(x) - \alpha I$.
This shows that f is strongly convex iff $\nabla f(x) - \alpha I$ is positive semi-definite.

4. Suppose f is positively homogeneous and convex. Then

$$f(x+y) = 2f(\frac{1}{2}x + \frac{1}{2}y) \leq 2(\frac{1}{2}f(x) + \frac{1}{2}f(y)) = f(x) + f(y)$$

Hence f is subadditive.
Suppose f is positively homogeneous and subadditive. Let $\lambda \in [0,1]$.

$$f(\lambda x + (1-\lambda)y) \leq f(\lambda x) + f((1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$$

Hence, f is convex.