Math4230 Exercise 3

1. Let C be a nonempty convex subset of $\mathbb{R}^n$. Let $f = (f_1, ..., f_m)$, where $f_i : C \to \mathbb{R},\ i = 1, ..., m$, are convex functions, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a convex function such that $g(u_1) \leq g(u_2)$, for all $u_1 \leq u_2$ in a convex set that contains $\{f(x)|x \in C\}$. Show that h defined by $h(x) = g(f(x))$ is convex over C. If in addition, $m = 1, g$ is strictly increasing and f is strictly convex, show that h is also strictly convex.

2. Show that the following functions are convex:
   (a) $f_1(x) = \ln(e^{x_1} + ... + e^{x_n}),$ where $x \in \mathbb{R}^n$.
   (b) $f_2(x) = ||x||^p$ with $p \geq 1$
   (c) $f_3(x) = e^{x^T Ax}$, where A is a positive semidefinite symmetric $n \times n$ matrix

3. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a differentiable function. We say that f is strongly convex with coefficient α if
   $$(\nabla f(x) - \nabla f(y))^T(x - y) \geq \alpha\|x - y\|^2, \forall x, y \in \mathbb{R}^n,$$
   where α is some positive scalar.
   (a) Show that if f is strongly convex with coefficient α, then f is strictly convex.
   (b) Assume that f is twice continuously differentiable. Show that strongly convexity of f with coefficient α is equivalent to the positive semi definiteness of $\nabla^2 f(x) - \alpha I$ for every $x \in \mathbb{R}^n$, where I is the identity matrix.

4. We say that $f : \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if $f(\alpha x) = \alpha f(x)$ for all $\alpha > 0$, and that f is *subadditive* if $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$. Show that a positively homogeneous function is convex if and only if it is subadditive.