Math4230 Exercise 2 Solution

1. (a) We may assume $\lambda_2 > 0$ ($\lambda_2 > 0$ is easy). Let $x \in C$. Since $0 \in C$, $\frac{\lambda_1}{\lambda_2}x = \frac{\lambda_1}{\lambda_2}x + \frac{\lambda_2-\lambda_1}{\lambda_2}0 \in C$.
   Then $\lambda_1 x = \lambda_2(\frac{\lambda_1}{\lambda_2}x) \in \lambda_2 C$.

   (b) We may assume $\alpha + \beta > 0$ ($\alpha + \beta = 0$ is easy). Then
   $$\alpha x + \beta y = (\alpha+\beta)(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y)$$

2. Suppose $\sum_{i=1}^m \lambda_i x_i + \lambda x = 0$ and $\sum_{i=1}^m \lambda_i + \lambda = 0$.
   Suppose $\lambda \neq 0$. Then
   $$-\sum_{i=1}^m \frac{\lambda_i}{\lambda} = 1$$
   So $x = -\sum_{i=1}^m \frac{\lambda_i}{\lambda} x_i \in \mathrm{aff}(\{x_1, ..., x_m\})$, which is a contradiction.
   Hence $\lambda = 0$. Then $\lambda_i = 0$ since $x_1, ..., x_m$ are affinely independent.
   Therefore, $x_1, ..., x_m, x$ are affinely independent.

3. Since $\{x_0, ..., x_m\} \subset \Delta_m \subseteq C$, $\mathrm{aff}(\{x_0, ..., x_m\}) \subseteq \mathrm{aff}(\Delta_m) \subseteq \mathrm{aff}(C)$.
   Since $x_0, ..., x_m$ are affinely independent, $\dim\{x_0, ..., x_m\} = m = \dim(C)$.
   Then $\mathrm{aff}(\{x_0, ..., x_m\}) = \mathrm{aff}(C)$ by dimension argument.
   Hence, $\mathrm{aff}(\{x_0, ..., x_m\}) = \mathrm{aff}(\Delta_m) = \mathrm{aff}(C)$

4. (a) Suppose $f(x), f(y) \leq a$. Let $\lambda \in [0, 1]$. Then
   $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda a + (1-\lambda)a = a.$$
   Hence $\lambda x + (1-\lambda)y \in \{x \in \mathbb{R}^n | f(x) \leq a\}$ and the level set is convex.

   (b) No. Consider $f(x) = x^2$. $C = (1, 2)$.