Math4230 Exercise 1 Solution

1. Suppose C is a convex. Let $x, y \in C$. Then since C is convex, $\frac{1}{2}x+\frac{1}{2}y \in C$. Since C is a cone, $2(\frac{1}{2}x + \frac{1}{2}y) = x + y \in C$.
   Conversely, suppose $C + C \subseteq C$. Let $\lambda \in (0, 1)$. Since C is a cone, then $\lambda x, (1 - \lambda)y \in C$, for all $x, y \in C$. Hence $\lambda x + (1 - \lambda)y \in C$. Therefore, C is convex.

2. (Interior) Let $x, y \in C^{\text{o}}$. Then there exists r such that balls with radius r centred at x and y are both inside C.
   Suppose $\alpha \in [0, 1]$ and $||z|| < r$. By convexity of C, we have,
   $$\alpha x + (1 - \alpha)y + z = \alpha(x + z) + (1 - \alpha)(y + z) \in C$$
   Therefore, $\alpha x + (1 - \alpha)y \in C^{\text{o}}$. Hence C^{o} is convex.
   (Closure) Let $x, y \in \overline{C}$. Then there exists sequences $\{x_k\} \subset C, \{y_k\} \subset C$ such that $x_k \to x, y_k \to y$. Suppose $\alpha \in [0, 1]$. Then for each k,
   $$\alpha x_k + (1 - \alpha)y_k \in C$$
   But $\alpha x_k + (1 - \alpha)y_k \to \alpha x + (1 - \alpha)y \in \overline{C}$. Hence, $\overline{C}$ is convex.

3. Let T be a linear map.
   (a) Let $y_1 = T(x_1), y_2 = T(x_2) \in f(C),\ \lambda \in [0, 1]$, where $x_1, x_2 \in C$.
   Then $\lambda y_1 + (1 - \lambda)y_2 = T(\lambda x_1 + (1 - \lambda)x_2)$.
   Since C is convex, $\lambda x_1+(1-\lambda)x_2 \in C$. Hence $\lambda y_1+(1-\lambda)y_2 \in T(C)$.
   (b) Let $x_1, x_2 \in T^{-1}(C'),\ \lambda \in [0, 1]$.
   Then $T(\lambda x_1 + (1 - \lambda x_2)) = \lambda T(x_1) + (1 - \lambda)T(x_2)$.
   Since $T(x_1), T(x_2) \in C'$ and C' is convex, $\lambda T(x_1)+(1-\lambda)T(x_2) \in C'$.
   Hence, $T^{-1}(C')$ is also convex.

4. (a) Let $y_1, y_2 \in f(C)$. Then $y_1 = \frac{u_1}{t_1},\ y_2 = \frac{u_2}{t_2}$.
   Let $\lambda \in [0, 1]$. Consider $\lambda\frac{u_1}{t_1} + (1 - \lambda)\frac{u_2}{t_2}$.
   We need to find α such that
   $$\lambda\frac{u_1}{t_1} + (1 - \lambda)\frac{u_2}{t_2} = \frac{\alpha u_1 + (1 - \alpha)u_2}{\alpha t_1 + (1 - \alpha)t_2} = f(\alpha(u_1, t_1) + (1 - \alpha)(u_2, t_2))$$
   It can be verified that $\alpha = \frac{\lambda t_2}{(1-\lambda)t_1+\lambda t_2}$ satisfies the above equation.
   This shows that $\lambda y_1 + (1 - \lambda)y_2 \in f(C)$

   (b) Let $(x_1, t_1), (x_2, t_2) \in f^{-1}(C)$. Let $\lambda \in [0, 1]$.
   We need to show that $\frac{\lambda x_1+(1-\lambda)x_2}{\lambda t_1+(1-\lambda)t_2} \in C$.
   Consider $\alpha = \frac{\lambda t_1}{\lambda t_1+(1-\lambda)t_2}$. Then
   $$\frac{\lambda x_1 + (1 - \lambda)x_2}{\lambda t_1 + (1 - \lambda)t_2} = \alpha\frac{x_1}{t_1} + (1 - \alpha)\frac{x_2}{t_2} \in C$$

(c) Consider

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

Since g is an affine function, it is convex.
Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^m$ be a perspective function. Then

$$\frac{Ax + b}{c^T x + d} = f(g(x))$$

Then $h(C)$ is convex if C is convex.