Math4230 Exercise 10 Solution

1. Suppose x^*, λ^* satisfy the KKT conditions. Then

$$\begin{aligned}\langle \nabla f(x^*), x - x^* \rangle &= \langle -\sum \lambda_i^* \nabla g_i(x^*), x - x^* \rangle \\ &\geq \sum \lambda_i^* (g_i(x^*) - g_i(x)) \\ &= -\sum \lambda_1^* g_i(x) \\ &\geq 0\end{aligned}$$

   The first inequality holds since g_i are convex. The second inequality holds since x is feasible and $\lambda_i^* \geq 0$.

2. (a) $2x + 3y + 2z$ is continuous and K is compact, hence an optimal solution exists.

   (b) We minimize $-2x - 3y - 2z$. KKT conditions:

$$\begin{aligned}(x^*)^2 + (y^*)^2 + (z^*)^2 &= 1, \\ x^* + y^* + z^* &\geq 0, \\ \lambda^*(x^* + y^* + z^*) &= 0, \\ \lambda^* &\geq 0, \\ -2 - \lambda^* + \mu^* 2x^* &= 0 \\ -3 - \lambda^* + \mu^* 2y^* &= 0 \\ -2 - \lambda^* + \mu^* 2z^* &= 0\end{aligned}$$

   (c) Adding the last 3 equations, we have $2\mu^*(x^* + y^* + z^*) = 3\lambda^* + 7$.
   If $x^* + y^* + z^* = 0$, then $\lambda^* = -7/3$. Contradiction.
   Hence $\lambda^* = 0$. So, $x^* = 1/\mu^*,\ y^* = 3/2\mu^*,\ z^* = 1/\mu^*$.
   But $(x^*)^2 + (y^*)^2 + (z^*)^2 = 1$, so $\mu^* = \sqrt{17}/2$.
   Hence, $x^* = 2\sqrt{17}/17,\ y^* = 3\sqrt{17}/17,\ z^* = 2\sqrt{17}/17$.

3. (a) Note that $\|x\| = 1$ is equivalent to $\|x\|^2 - 1 = 0$. We use this as constraint.
   KKT conditions:

$$\begin{aligned}\|x^*\|^2 &= 1 \\ 2Ax^* + 2\mu^* x^* &= 0\end{aligned}$$

   (b) Assuming the KKT conditions are necessary, we have $Ax^* = -\mu^* x^*$.
   Therefore, x^* is an eigenvetor of A with eigenvalue $-\mu^*$.
   Since $\langle x^*, Ax^* \rangle = -\mu^* \|x\|^2 = -\mu^*$, the optimal value is an eigenvalue of A.

4. (a) Note that 0 is the only feasible point. Hence the optimal value is 0.
   $L(x, \lambda) = x + \lambda x^2$.
   So the dual function $g(\lambda) = -1/4\lambda$ if $\lambda > 0$. ($g(\lambda) = -\infty$ if $\lambda \leq 0$.)
   Dual Problem

$$\max_{\lambda \geq 0} -1/4\lambda$$

The dual optimal value is hence 0. Therefore, there is no duality gap.

(b) There is no λ such that $-1/4\lambda = 0$. Hence there is no dual optimal solution.
(This example shows that dual optimal solution may not exist, even if there is no duality gap.)