Math4230 Exercise 10 Solution

1. (a) Feasible set: $\{x|\ 2 \le x \le 4\}$.
   Optimal value: 5 Optimal solution: $x = 2$.

   (b) $L(x,\lambda) = x^2 + 1 + \lambda(x-2)(x-4) = (1+\lambda)x^2 - 6\lambda x + (1+8\lambda)$.
   For $\lambda > -1, L(x,\lambda)$ reaches minimum at $x = \frac{3\lambda}{1+\lambda}$.
   For $\lambda \le -1, L(x,\lambda)$ is unbounded below. So

$$g(\lambda) = \begin{cases} \frac{-9\lambda^2}{1+\lambda} + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \le -1 \end{cases}$$

   (c) The dual problem is

$$\max_{\lambda \ge 0} \frac{-9\lambda^2}{1+\lambda} + 1 + 8\lambda$$

   So the dual optimal value is 5 and the dual optimal solution is $\lambda = 2$. Hence strong duality holds. (This is guaranteed since the Slater's condition holds)

2. $L(x,\lambda) = \langle c, x\rangle + \lambda f(x)$. So dual function

$$g(\lambda) = \inf \langle c, x\rangle + \lambda f(x) = -\sup(-\langle c, x\rangle - \lambda f(x))$$

   If $\lambda = 0$, then $g(\lambda)$ is unbounded below.
   If $\lambda > 0$, then

$$g(\lambda) = -\lambda \sup(-\langle c/\lambda, x\rangle - f(x)) = -\lambda f^*(-c/\lambda)$$

   Hence the dual problem is

$$\max_{\lambda > 0} -\lambda f^*(-c/\lambda)$$

3. (a) Feasible set: $\{(1,0)\}$
   Optimal solution: $x^* = (1,0)$. Optimal value: 1

   (b) KKT conditions:

$$\begin{aligned}
(x_1^* - 1)^2 + (x_2^* - 1)^2 &\le 1, \\
(x_1^* - 1)^2 + (x_2^* + 1)^2 &\le 1, \\
\lambda_1^*((x_1^* - 1)^2 + (x_2^* - 1)^2 - 1) &= 0, \\
\lambda_2^*((x_1^* - 1)^2 + (x_2^* + 1)^2 - 1) &= 0, \\
x_1^* + \lambda_1^*(x_1^* - 1) + \lambda_2^*(x_1^* - 1) &= 0, \\
x_2^* + \lambda_1^*(x_2^* - 1) + \lambda_2^*(x_2^* + 1) &= 0
\end{aligned}$$

   Since $x_1^* = 1, x_2^* = 0$, we have

$$1 + \lambda_1^*(1-1) + \lambda_2^*(1-1) = 0$$

   Hence, there is no λ_1^*, λ_2^* that satisfy the above equation.